Enumerative Combinatorics Homework 6

Mikhail Lavrov

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1. The Fibonacci sequence satisfies the identity $\sum_{k=0}^{n} F_k = F_{n+2} - 1$. (For example, if we take $n = 5$, then the sum $F_0 + F_1 + F_2 + F_3 + F_4 + F_5$ is $0+1+1+2+3+5 = 12$, which is equal to $F_7 - 1$ or $13 - 1$.)

   Prove this identity by induction on n. (We might see other ways to prove this identity in class; I haven't decided yet.)

2. For $n \geq 1$, let t_n be the number of n-bit sequences (that is, sequences of n 0's and 1's) that do not have three of the same symbol in a row. (For example, $t_3 = 6$, counting the sequences 001, 010, 011, 100, 101, and 110; the sequences 000 and 111 are not counted.)

   What is the relationship between this sequence and the sequence of Fibonacci numbers? Either give a combinatorial argument for the relationship, *or* prove it by induction.

3. Let $A(x)$ be the OGF for a sequence $a_0, a_1, a_2, \dots$ beginning with $a_0 = 0$ and $a_1 = 1$.

   (a) Find the generating function $B(x)$ for the sequence $b_0, b_1, b_2, \dots$ satisfying $b_n = 2^n - a_{n-2}$; express your answer in terms of $A(x)$.

   (b) Suppose that a_n satisfies the recurrence relation $a_n = 2^n - a_{n-2}$. Use this information, and your work in part (a), to solve for $A(x)$.

4. In each part, find the first 10 terms of the sequence with the given generating function:

   (a) $(2+x)^3$.

   (b) $\frac{3}{1-4x^3}$.

   (c) $\frac{2}{1-2x} - \frac{1}{1+2x} + 2x$.

   (d) $\frac{1}{(1-2x)^2}$.

5. A *magic missile* is a glowing dart of magical force that unerringly strikes your enemy for 2, 3, 4, or 5 damage. Find generating functions in which the coefficient of x^n is the number of ways in which you can deal a total of n damage:

   (a) With just one magic missile.

   (b) With three magic missiles, launched one after the other. (Order matters. For example, you can deal 7 damage in 3 ways, by dealing $2+2+3$, $2+3+2$, or $3+2+2$ damage.)

   (c) With any number of magic missiles, launched one after the other.