Enumerative Combinatorics Homework 7

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1. How many ways are there to distribute 32 pieces of candy to 12 children so that each child gets either 2 or 3 pieces of candy?

2. In this problem, you will find generating functions for the number of ways to put plain marbles into labeled boxes, under various constraints:

   (a) There are five boxes, and each box must contain an odd number of marbles. The coefficient of x^k should be the number of ways to distribute k marbles.

   (b) There are five boxes, numbered 1 through 5, and the number on a box is the maximum number of marbles it can contain. The coefficient of x^k should be the number of ways to distribute k marbles.

   (c) There are five boxes, and each box can contain 0, 2, or 5 marbles. The coefficient of x^k should be the number of ways to distribute *up to* k marbles: k marbles or fewer.

3. Once again, consider paths from $(0, 0)$ to (n, n) in an $n \times n$ grid. How many of these paths:

   - never go below the line $y = x$, **and**
   - aside from $(0, 0)$ and (n, n), never touch that line either?

   (For a point of extra credit, solve the original version of the problem. This was to count the paths that visit the line $y = x$ exactly three times—at $(0, 0)$, at (n, n), and at some unspecified intermediate point.)

4. (a) You have n beads, all of different colors; assume $n \geq 3$. How many ways are there to put them on a bracelet, up to symmetry?

   (A bracelet has two symmetries: all rotations of a pattern are equivalent, and we can also flip a bracelet over, reversing the pattern.)

   (b) Write down an exponential generating function for your answer to (a). Simplify as much as possible. You can have your EGF do anything you like for the $n = 0, n = 1, n = 2$ cases, whatever is convenient.

   If you know a Taylor series for natural logarithm, I encourage you to use it to give your answer a closed form; otherwise, leave it as an infinite sum.

5. (a) Find the EGF for the sequence beginning

   $$0, 1, 4, 12, 32, 80, 192, 448, 1024, 2304, 5120, \dots$$

   whose n^{th} term is $n2^{n-1}$.

   (b) Find the first 10 terms of the sequence whose EGF is $(x + \frac{1}{2}x^2)e^x$.