Enumerative Combinatorics Homework 8

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1 Course evaluations

Course evaluations are now available! I am pretty sure you can find a link to them on the D2L home page (`https://kennesaw.view.usg.edu/d2l/home`); let me know if you have trouble. The deadline to fill out the course evaluations is by the end of the day on Monday, April 29.

I really appreciate getting feedback on my teaching, so it means a lot to me if you fill these out. (That's especially true in this class, which I'm teaching for the first time!) As added incentive, if half of you fill out a course evaluation, I will wear a funny hat for the final exam.

2 Homework problems

1. Let $\mathcal{A}$ be the set of all strings that can be written using each of the letters in "PROBLEM" at most once, in any order:

$$\mathcal{A} = \{\varepsilon, \mathsf{P}, \mathsf{R}, \mathsf{O}, \dots, \mathsf{POEM}, \dots, \mathsf{PROBLEM}, \dots, \mathsf{BELMOPR}\}.$$

   Let $\mathcal{B}$ be the the set of all strings consisting of 0 or more Z's:

$$\mathcal{B} = \{\varepsilon, \mathsf{Z}, \mathsf{ZZ}, \mathsf{ZZZ}, \dots\}.$$

   Find an exponential generating function in which the coefficient of $\frac{x^n}{n!}$ is the number of strings of length n obtained by interlacing a string from $\mathcal{A}$ and a string from $\mathcal{B}$.

2. Find a closed-form solution to the following recurrence relations:
   (a) $a_0 = 3$, $a_1 = \frac{1}{2}$, and $a_n = a_{n-1} + \frac{3}{4}a_{n-2}$ for $n \geq 2$.
   (b) $b_0 = 1$, $b_1 = 1$, and $b_n = b_{n-1} - \frac{1}{4}b_{n-2}$ for $n \geq 2$.
   (c) $c_0 = \frac{1}{3}$ and $c_n = 2c_{n-1} + (-1)^n$ for $n \geq 1$.

3. (a) How many ways are there to permute the letters of "SASSAFRAS"?
   (b) How many of them do not begin with an "S"?
   (c) How many of them neither begin nor end with an "S"?

4. **Review of the twelvefold way.** Identify each of these problems as the appropriate case of the twelvefold way. (Feel free to use either the language of functions, or the language of putting marbles in boxes, whichever you're more comfortable with.)

   Then, find a numerical answer. (Here, I would like you to evaluate everything until you get a number, but you should feel free to use a calculator if you like.)

   (a) How many 7-digit phone numbers use every digit at most once?

   (b) How many 7-digit phone numbers use only the digits $1, 2, 3, 4, 5$ and use each of them at least once?

   (c) How many 7-digit phone numbers have their digits sorted in increasing order? (Repeated digits are allowed: the phone number "555-6788" would count.)

5. **Review of bijections.**

   (a) Find a bijection between the set of 2-element subsets of $\{1, 2, 3, 4, 5, 6\}$ and the set of 4-element subsets of $\{1, 2, 3, 4, 5, 6\}$.

   (b) Find a bijection between the set of all 8-bit strings that begin and end with the same symbol (like 01001110) and the set of all subsets of $\{1, 2, 3, 4, 5, 6, 7\}$.

   (c) Find a bijection between the set of all 3-digit numbers and the set of all 5-digit palindromes. (A palindrome is a number like "20002" that reads the same forwards and backwards.)