Probability Theory Homework 4

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1. Consider the following random experiment: you roll two dice whose faces have the labels ▢, ▢, ⚀, ⚀, ⚁, ⚁: that is, when you roll one die, you get a result of 0, 1, or 2 with equal probability.

   (a) Write down the "big" sample space for this random experiment: the one which gives us the results of both rolls as the outcome. (There are two plausible ways to do this; pick the one where we sample uniformly from this sample space.)

   (b) Let $\mathbf{X}$ be the random variable which gives the larger of the two numbers rolled. For example, if the two rolls are ▢ and ⚀, then $\mathbf{X} = 1$.

   Describe $\mathbf{X}$ as a function from the sample space to its range $R_{\mathbf{X}} = \{0, 1, 2\}$. (One description I'd be happy with is an arrow diagram, though you may choose anything else as long as you convey the same information.)

   (c) Write down the probability mass function $P_{\mathbf{X}}(k)$.

2. A random variable $\mathbf{X}$ has range $R_{\mathbf{X}} = \{1, 2, 3, 4, 5, 6\}$ and probability mass function $P_{\mathbf{X}} \colon R_{\mathbf{X}} \to [0, 1]$ given by

$$P_{\mathbf{X}}(k) = \begin{cases} c & k \in \{1, 2\}, \\ 2c & k \in \{3, 4, 5, 6\}. \end{cases}$$

   (Watch out for a common mistake: this piecewise definition says that $P_{\mathbf{X}}(1) = c$ and that $P_{\mathbf{X}}(2) = c$, not that $\Pr[\mathbf{X} \in \{1, 2\}] = c$.)

   (a) Find the value of c for which this is a valid probability mass function.

   (b) Find $\Pr[\mathbf{X} \geq 4]$.

   (c) Find the expected value $\mathbb{E}[\mathbf{X}]$.

3. The card game Hearts is played with a standard 52-card deck. There are 13 hearts in the deck, each of which is worth 1 point. There is also one card called the Queen of Spades in the deck, which is worth 13 points.

   You draw a hand of 13 cards (without replacement).

   (a) The number of hearts you draw follows a Hypergeometric distribution. What is its expected value?

(b) The number of Queens of Spades is, of course, either 0 or 1 (since there's only one in the deck), so it follows a Bernoulli distribution. What is its expected value?

(c) What is the expected value of the total number of points in the hand you draw?

4. A random variable $\mathbf{W}$ has range $R_{\mathbf{W}} = \{1, 2, 3, \dots\}$ and the probability mass function given by

$$P_{\mathbf{W}}(k) = \frac{1}{k2^k \ln 2}$$

for all $k \in R_{\mathbf{W}}$. Find the expected value $\mathbb{E}[\mathbf{W}]$.

5. Each of the following random variables has a Binomial, Geometric, Pascal, Hypergeometric, or Poisson distribution. Identify the distribution, and give its parameters.

(a) You draw a hand of 5 cards from a standard 52-card deck. $\mathbf{A}$ is the number of aces you draw. (There are 4 aces in the deck.)

(b) Bird watchers in a large urban park report an average of 1.5 sightings of a rare bird species per day; $\mathbf{B}$ is the number of bird sightings of that rare bird species on one particular day.

(c) You receive many emails every day, but 90% of them are junk emails. (Let's assume that you don't have a spam filter to catch these junk emails.) Today, you receive 15 emails; $\mathbf{J}$ is the number of them that are junk.

(d) Each booster pack of the famous card game *Sorcery: the Collecting* contains one mega-rare card, chosen uniformly from the 10 mega-rare cards in the set. Your goal is to get a full playset of 4 copies of the mega-rare card Purple Cabbage.

You buy and open booster packs one at a time; $\mathbf{P}$ is the number of booster packs you will have to open in order to get four copies of Purple Cabbage.