Probability Theory Homework 8

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due **Monday**, April 29, 2024

1 Course evaluations

Course evaluations are now available! I am pretty sure you can find a link to them on the D2L home page (`https://kennesaw.view.usg.edu/d2l/home`); let me know if you have trouble. The deadline to fill out the course evaluations is by the end of the day on Monday, April 29; the same due date as this homework assignment.

I really appreciate getting feedback on my teaching, so it means a lot to me if you fill these out. As added incentive, if at least half of you fill out a course evaluation, I will wear a funny hat for the final exam.

2 Problems on the last few lectures

1. Just to give you a break from all the integrals, here is a mixture distribution problem about two discrete random variables.

   Suppose I have *three* dice in my pocket: two fair dice that land on 6 with probability $\frac{1}{6}$, and a loaded die that lands on 6 with probability $\frac{1}{2}$. I take a random die out of my pocket, roll it 10 times, and count the number of sixes rolled. Let $\mathbf{N}$ be the result I get.

   (a) Describe the distribution of $\mathbf{N}$ as a mixture of two distributions we know, and give the weights used in the mixture.

   (b) What is $\mathbb{E}[\mathbf{N}]$?

   (c) What is $\Pr[\mathbf{N} = 0]$?

2. Suppose that two random variables $\mathbf{X}$ and $\mathbf{Y}$ have the joint PDF

$$f_{\mathbf{XY}}(u, v) = \begin{cases} 60u^2v & u \geq 0, v \geq 0, \text{ and } u + v \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Are $\mathbf{X}$ and $\mathbf{Y}$ independent?

   (b) What is the marginal distribution $f_{\mathbf{X}}(t)$?

(c) What is $\Pr[\mathbf{X} \geq \mathbf{Y}]$? *(To set up the right integral, it might help you to draw the range of* $(\mathbf{X}, \mathbf{Y})$ *in the uv-plane and identify the region within that range where* $u \geq v$.*)*

3. You are a birdwatcher listening for bird calls in a quiet park on a peaceful April day.

   Based on your experience, you hear an average of one bird call per minute. Also, when you hear a bird call, it's a robin $\frac{1}{2}$ of the time, a woodpecker $\frac{1}{3}$ of the time, and a sparrow $\frac{1}{6}$ of the time.

   Assuming that three types of bird calls are independent Poisson processes, identify the distributions of the following random variables: give the name of the distribution and its parameters.

   (a) $\mathbf{W}$ is the time it takes you to hear a woodpecker call.

   (b) $\mathbf{R}$ is the number of robin calls you hear over the course of an hour.

   (c) $\mathbf{T}$ is the time it takes you to hear 10 bird calls total.

   (d) After you've heard 10 bird calls total, $\mathbf{S}$ is the number of sparrow calls you've heard.

3 Review problems

4. An arcade machine costs one dollar to play. Every time you play, there is a $\frac{1}{10}$ chance of winning.

   (a) Let $\mathbf{A}$ be the number of times you play until you win. Find the variance $\mathrm{Var}[\mathbf{A}]$.

   (b) Ten people play the arcade machine; each one keeps playing until they win, and then it's the next person's turn. Let $\mathbf{B}$ be the number of times they play in total. Find the variance $\mathrm{Var}[\mathbf{B}]$.

   (c) Ten years later, you come back to the arcade machine, and it's exactly the same, except that due to inflation, the machine costs $10 to play.

   Let $\mathbf{C}$ be the the total cost of playing until you win. Find the variance $\mathrm{Var}[\mathbf{C}]$.

5. Let $\mathbf{D}$ be a random variable with range $\{1, 2, 3, 4\}$ and the probability mass function

$$P_{\mathbf{D}}(1) = 0.1, \qquad P_{\mathbf{D}}(2) = 0.2, \qquad P_{\mathbf{D}}(3) = 0.3, \qquad P_{\mathbf{D}}(4) = 0.4.$$

   If you take 10 independent random samples from the distribution of $\mathbf{D}$, what is the probability that you get the results $1, 2, 2, 3, 3, 3, 4, 4, 4, 4$, in any order?