IMPROVED CHEN-RICCI INEQUALITY FOR LAGRANGIAN SUBMANIFOLDS IN QUATERNION SPACE FORMS

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Abstract. In this article, we obtain an improved Chen-Ricci inequality and completely classify Lagrangian submanifolds in quaternion space forms satisfying the equality. Our result is an affirmative answer to Problem 4.6 in [12].

1. Introduction

Let $M$ be a Riemannian $n$-manifold and $X$ be a unit vector. We choose an orthonormal frame $\{e_1, \ldots, e_n\}$ in $T_xM$ such that $e_1 = X$. We denote the Ricci curvature at $X$ by

$$Ric(X) = K_{12} + \cdots + K_{1n},$$

where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $e_i, e_j$.

In [1] B.-Y. Chen proved the following Chen-Ricci inequality on Ricci curvature for any $n$-dimensional submanifold in Riemannian manifold of constant sectional curvature $c$:

$$Ric(X) \leq n \left( \frac{c}{2} + \frac{n^2}{4} ||H||^2 \right).$$

This inequality is not optimal for Lagrangian submanifolds in complex space forms. Using an optimization technique, Oprea in [10] (also see [11]) proved

$$Ric(X) \leq n \left( \frac{c}{4} + n ||H||^2 \right),$$

which improves the Chen-Ricci inequality for Lagrangian submanifolds in complex space forms of constant holomorphic sectional curvature $c$.

In [5] we provided an algebraic proof for the improved Chen-Ricci inequality and completely characterized Lagrangian submanifolds in complex space forms satisfying the equality. In this article, we extend the improved Chen-Ricci inequality to Lagrangian submanifolds in quaternion space forms. We also provide a detailed affirmative answer to Problem 4.6 in [12], completing the remark 3.2 in [5].

Theorem 3.1 and Corollary 3.2 improve a number of results in [1],[5],[7] and [8] for Lagrangian submanifolds in quaternion space forms.
2. Preliminaries

Let $\tilde{M}^n$ be a 4n-dimensional Riemannian manifold with metric $g$. $\tilde{M}^n$ is called quaternion Kaehler manifold if there exists a 3-dimensional vector bundle $V$ of tensors of type (1,1) over $\tilde{M}^n$ with local basis of almost Hermitian structures $I, J, K$ and $K$ such that

(a) $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$, $I^2 = J^2 = K^2 = -I$;

(b) for any local cross-section $\eta$ of $V$, $\nabla_X\eta$ is also a cross-section of $V$, where $X$ is an arbitrary vector field on $\tilde{M}^n$ and $\nabla$ the Riemannian connection on $\tilde{M}^n$.

In fact, condition (b) is equivalent to the following condition:

(b') there exist local 1-forms $p, q$ and $r$ such that

\[
\begin{align*}
\nabla_X I &= r(X)J - q(X)K \\
\nabla_X J &= -r(X)I + p(X)K \\
\nabla_X K &= q(X)I - p(X)J.
\end{align*}
\]

Now let $X$ be a unit vector on $\tilde{M}^n$. Then $X, IX, JX$ and $KX$ form an orthonormal frame on $\tilde{M}^n$. We denote by $Q(X)$ the 4-plane spanned by them. For any two orthonormal vectors $X, Y$ on $\tilde{M}^n$, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X, Y)$ spanned by $X, Y$ is called a totally real plane. Any 2-plane in a $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane $\pi$ is called the quaternionic sectional curvature of $\pi$. A quaternionic Kaehler manifold is a quaternion space form if its quaternionic sectional curvature are equal to a constant, say $c$. We denote such a 4n-dimensional quaternion space form by $\tilde{M}^n(c)$.

It is known that a quaternionic Kaehler manifold $M^n$ is a quaternion space form if and only if its curvature tensor $\tilde{R}$ is of the following form [6]:

\[
\tilde{R}(X, Y)Z = \frac{c}{4} (g(Y, Z)X - g(X, Z)Y + g(IY, Z)IX - g(IX, Z)IY + 2g(X, IY)IZ \\
+ g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ \\
+ g(KY, Z)KX - g(KX, Z)KY + 2g(X, KY)KZ).
\]

Let $f : M \to \tilde{M}^n$ be an isometric immersion of a Riemannian $n$-manifold $M$ into a 4n-dimensional quaternion space form $\tilde{M}^n(c)$. Then $M$ is called a Lagrangian (or totally real) submanifold if each 2-plane of $M$ is mapped into a toally real plane in $\tilde{M}^n(c)$.

From now on we assume that $M$ is a Lagrangian submanifold of a 4n-dimensional quaternion space form $\tilde{M}^n(c)$. The formulas of Gauss and Weingarten are given respectively by

\[
\begin{align*}
\nabla_X Y &= \nabla_X Y + h(X, Y), \\
\nabla_X \xi &= -A_\xi X + D_X \xi,
\end{align*}
\]

for tangent vector fields $X$ and $Y$ and normal vector fields $\xi$, where $D$ is the normal connection. The second fundamental form $h$ is related to $A_\xi$ by

\[
\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.
\]

The mean curvature vector $H$ of $M$ is defined by

\[
H = \frac{1}{n} \text{trace} h.
\]

We choose a local orthonormal frame field in $\tilde{M}^n(c)$:

\[
\begin{align*}
&e_1, e_2, \ldots, e_n; & e_{1(1)} = Ie_1, \ldots, e_{1(n)} = Ie_n; \\
&e_{J(1)} = Je_1, \ldots, e_{J(n)} = Je_n; & e_{K(1)} = Ke_1, \ldots, e_{K(n)} = Ke_n,
\end{align*}
\]

in such a way that, restricting to $M$, $e_1, \ldots, e_n$ are tangent to $M$.

We will use the following convention on the range of indices:
With the orthonormal frames from (2.2), we have

\begin{equation}
A, B, C, D = 1, \ldots, n, I(1), \ldots, I(n), J(1), \ldots, J(n), K(1), \ldots, K(n),
\end{equation}

\begin{equation}
i, j, k, l = 1, \ldots, n,
\end{equation}

\begin{equation}
\alpha, \beta = I(1), \ldots, I(n), J(1), \ldots, J(n), K(1), \ldots, K(n),
\end{equation}

\begin{equation}
\phi_1 = I, \ \phi_2 = J, \ \phi_3 = K,
\end{equation}

\begin{equation}
\phi_t(k) = I(k), \ \phi_t(k) = J(k), \ \phi_t(k) = K(k).
\end{equation}

We set \( h^\alpha_{ij} = g((e_i, e_j), e_\alpha) \). Then for any given \( r \) we have (see (2.9) in [4])

\begin{equation}
(2.3) \quad h^\phi_t(k) = h^\phi_t(\alpha) = h^\phi_t(\beta), \quad r = 1, 2, 3.
\end{equation}

Chen introduced the concept of \textit{Lagrangian H-umbilical submanifolds} in [2] to study the "simplest" Lagrangian submanifolds next to the totally geodesic ones. We can extend the notion of \textit{Lagrangian H-umbilical submanifolds} to Lagrangian submanifolds of a quaternion manifold ([9]). By a Lagrangian \( H \)-umbilical submanifold of a quaternion manifold \( M^n \) we mean a Lagrangian submanifold whose second fundamental form takes the following simple form:

\begin{equation}
\begin{aligned}
&h(e_1, e_1) = \lambda_1 I(e_1) + \lambda_2 J(e_1) + \lambda_3 K(e_1) \\
&h(e_2, e_2) = \mu_1 I(e_1) + \mu_2 J(e_1) + \mu_3 K(e_1), \\
&h(e_1, e_2) = 0, \quad j \neq k, \quad j, k = 2, \ldots, n
\end{aligned}
\end{equation}

for some suitable functions \( \lambda_r \) and \( \mu_r, r = 1, 2, 3 \) with respect to some suitable local orthonormal frame field.

3. Improved Chen-Ricci Inequality

We first take a look at the mean curvature vector \( H \). We set \( \phi_1 = I, \ \phi_2 = J, \ \phi_3 = K \) as in the previous section. With the orthonormal frames from (2.2), we have

\begin{equation}
\begin{aligned}
h(e_1, e_1) &= \sum_{\alpha = I(1)}^{K(n)} h^\alpha_{11} e_\alpha = h^{I(1)}_{11} e_{I(1)} + \cdots + h^{I(n)}_{11} e_{I(n)} + \\
&\quad + h^{J(1)}_{11} e_{J(1)} + \cdots + h^{J(n)}_{11} e_{J(n)} + \\
&\quad + h^{K(1)}_{11} e_{K(1)} + \cdots + h^{K(n)}_{11} e_{K(n)} \\
&= \sum_{r=1}^{3} \sum_{k=1}^{n} h^\phi_t(k) e_{\phi_t(k)}.
\end{aligned}
\end{equation}

Similarly,

\begin{equation}
\begin{aligned}
h(e_i, e_i) &= \sum_{r=1}^{3} \sum_{k=1}^{n} h^\phi_t(k) e_{\phi_t(k)}.
\end{aligned}
\end{equation}

We set

\begin{equation}
(3.1) \quad H^r_t = \frac{1}{n} \sum_{k=1}^{n} h^\phi_t(k), r = 1, 2, 3.
\end{equation}

Then
Theorem 3.1. Let $M$ be a Lagrangian submanifold of real dimension $n$ ($n \geq 2$) in a 4-dimensional quaternion space form $\tilde{M}^n(c)$, $x$ a point in $M$ and $X$ a unit tangent vector in $T_xM$. Then we have

\[
\|H\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{3} (H^i_k)^2.
\]

(3.3)

The equality sign holds for any unit tangent vector at $x$ if and only if either

(i) $x$ is a totally geodesic point or

(ii) $n = 2$ and $x$ is an $H$-umbilical point with $\lambda_r = 3\mu_r, r = 1,2,3$.

Proof. We fix the point $x$ in $M$. Let $X$ be any unit tangent vector at $x$. We choose an orthonormal frame $e_1, \ldots, e_n, f(e_1), \ldots, K(e_n)$ such that $e_1, \ldots, e_n$ are tangent to $M$ at $x$ with $e_1 = X$. From Gauss equation we have

\[
\tilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - g(h(e_1, e_1), h(e_j, e_j)) + g(h(e_1, e_j), h(e_1, e_j))
\]

or

\[
\tilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - \sum_{r=1}^{3} \sum_{k=1}^{n} (h^j_{1r}(k) h^j_{kr}(k) - (h^j_{1r}(k))^2), \forall j \in 2,n.
\]

Hence we have

\[
(n - 1) \frac{c}{4} = \tilde{Ric}(X) - \sum_{r=1}^{3} \sum_{k=1}^{n} (h^j_{1r}(k) h^j_{kr}(k) - (h^j_{1r}(k))^2).
\]

Therefore
Using (2.3), we have
\[
Ric(X) - (n - 1) \frac{c}{4} \leq \sum_{r=1}^{3} \sum_{k=1}^{n} \sum_{j=2}^{n} (h^{\phi_r(k)}_{ij} h^{\phi_r(k)}_{jj} - (h^{\phi_r(k)}_{ij})^2)
\]
\[
\leq \sum_{r=1}^{3} \sum_{k=1}^{n} \sum_{j=1}^{n} h^{\phi_r(k)}_{ij} h^{\phi_r(k)}_{jj} - \sum_{r=1}^{3} \sum_{j=2}^{n} (h^{\phi_r(1)}_{ij})^2 - \sum_{r=1}^{3} \sum_{j=2}^{n} (h^{\phi_r(j)}_{ij})^2.
\]

Using (2.3), we have
\[
Ric(X) - (n - 1) \frac{c}{4} \leq (\sum_{r=1}^{3} \sum_{k=1}^{n} h^{\phi_r(k)}_{ij} h^{\phi_r(k)}_{jj} - \sum_{r=1}^{3} \sum_{j=2}^{n} (h^{\phi_r(1)}_{ij})^2 - \sum_{r=1}^{3} \sum_{j=2}^{n} (h^{\phi_r(j)}_{ij})^2),
\]
or
\[
Ric(X) - (n - 1) \frac{c}{4} \leq \sum_{r=1}^{3} \sum_{k=2}^{n} h^{\phi_r(1)}_{ij} h^{\phi_r(k)}_{jj} - \sum_{r=1}^{3} \sum_{j=2}^{n} (h^{\phi_r(1)}_{ij})^2 + h^{\phi_r(1)}_{ij} h^{\phi_r(k)}_{jj} - \sum_{r=1}^{3} \sum_{j=2}^{n} (h^{\phi_r(1)}_{ij})^2.
\]

From Cauchy-Schwarz’s inequality and (3.2), we have
\[
(h^{\phi_r(1)}_{ij} - \frac{n}{2} H_r)^2 + \sum_{j=2}^{n} (h^{\phi_r(1)}_{ij})^2 \geq \frac{1}{n} \left( \frac{1}{2} h^{\phi_r(1)}_{11} + \frac{1}{2} h^{\phi_r(1)}_{22} + \cdots + \frac{1}{2} h^{\phi_r(1)}_{nn} \right)^2 = \frac{n}{4} (H_r^1)^2, \quad r = 1, 2, 3,
\]
or equivalently
\[
\sum_{j=2}^{n} (h^{\phi_r(1)}_{jj})^2 - h^{\phi_r(1)}_{ij} \sum_{j=2}^{n} h^{\phi_r(1)}_{jj} \geq \frac{n(1 - n)}{4} (H_r^1)^2, \quad r = 1, 2, 3.
\]

Similarly, by Cauchy-Schwarz’s inequality, we have
\[
(h^{\phi_r(k)}_{ij})^2 + \frac{n}{2} H_r^k - h^{\phi_r(k)}_{ij} \sum_{j=2}^{n} h^{\phi_r(k)}_{jj} \geq \frac{n^2}{8} (H_r^k)^2, \quad r = 1, 2, 3,
\]
which is equivalent to
\[
(h^{\phi_r(k)}_{ij})^2 - h^{\phi_r(k)}_{ij} \sum_{j=2}^{n} h^{\phi_r(k)}_{jj} \geq -\frac{n^2}{8} (H_r^k)^2, \quad r = 1, 2, 3.
\]

From (3.6), (3.7) and (3.8), we have
\[
Ric(X) - \frac{n - 1}{4} c \leq \sum_{r=1}^{3} \sum_{k=2}^{n} \left( \frac{n^2}{8} (H_r^k)^2 + \frac{n(n - 1)}{4} (H_r^1)^2 \right)
\]
\[
\leq \frac{n(n - 1)}{4} \sum_{r=1}^{3} \sum_{k=2}^{n} (H_r^k)^2 + (H_r^1)^2)
\]
\[
= \frac{n(n - 1)}{4} ||H||^2,
\]
which implies (3.4).

Now assume the equality sign of (3.4) holds for any unit tangent vector $X$ at $x$. Inequalities in (3.5), (3.7) and (3.8) become equalities. Thus, we have
\[
h^{\phi_r(1)}_{jk} = 0, \quad \forall j, k \geq 2, j \neq k, \quad r = 1, 2, 3,
\]
we get

\[ 2h_{\phi_r}^{(1)} - nH_1^1 = 2h_{\phi_r}^{(2)} = \cdots = 2h_{\phi_n}^{(1)}, \quad r = 1, 2, 3, \]

\[ 4h_{\phi_r}^{(k)} = nH_r^k, \quad k = 2, \ldots, n, \quad r = 1, 2, 3. \]

Also, by (3.9), we have either

(1) \( n \geq 3 \) and \( H_2^2 = H_2^3 = \cdots = H_n^2 = 0, \quad r = 1, 2, 3 \)

(2) \( n = 2 \).

Case (1) \( n \geq 3 \). We have \( H_2^2 = H_2^3 = \cdots = H_n^2 = 0, \quad r = 1, 2, 3 \). From (3.12) we have

\[ h_{\phi_r}^{(1)} = h_{\phi_r}^{(2)} = \frac{nH_r^2}{4} = 0, \quad \forall j \geq 2, \quad r = 1, 2, 3. \]

From this and (3.10) and (3.11), \( (h_{\phi_r}^{(1)}) \) must be diagonal with \( h_{\phi_r}^{(1)} = (n + 1)h_{\phi_2}^{(1)} \)
and \( h_{\phi_r}^{(2)} = \frac{1}{2}H_r^1, \quad \forall j \geq 2, \quad r = 1, 2, 3. \)

Now if we compute \( \text{Ric}(e_2) \) as we do for \( \text{Ric}(X) = \text{Ric}(e_1) \) in (3.5), from the equality we get

\[ h_{\phi_r}^{(2)} = h_{\phi_r}^{(2)} = 0, \quad \forall k \neq 2, \quad j \neq 2, \quad k \neq j, \quad r = 1, 2, 3. \]

From the equality and (3.11), we get

\[ h_{\phi_r}^{(2)} = h_{\phi_r}^{(2)} = \frac{H_r^2}{2} = 0, \quad r = 1, 2, 3. \]

Since the equality holds for all unit tangent vector, the argument is also true for matrices \( (h_{\phi_r}^{(1)}) \). Now finally \( h_{\phi_r}^{(2)} = h_{\phi_r}^{(2)} = \frac{H_r^2}{2} = 0, \quad \forall j \geq 3, \quad r = 1, 2, 3. \)

Therefore matrix \( (h_{\phi_r}^{(2)}) \) has only two possible nonzero entries (i.e. \( h_{\phi_r}^{(2)} = h_{\phi_r}^{(2)} = h_{\phi_2}^{(1)} = \frac{H_2^2}{2}, \quad r = 1, 2, 3 \)). Similarly matrix \( (h_{\phi_r}^{(1)}) \) has only two possible nonzero entries

\[ h_{\phi_r}^{(1)} = h_{\phi_r}^{(1)} = h_{\phi_2}^{(1)} = \frac{H_2^1}{2}, \quad \forall j \geq 3, \quad r = 1, 2, 3. \]

We now compute \( \text{Ric}(e_2) \) as follows:

\[ \tilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - g(h(e_2, e_2), h(e_j, e_j)) + (h(e_2, e_j), h(e_2, e_j)), \]

so we have

\[ \tilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \frac{1}{2} \left( \sum_{r=1}^{n} \left( \frac{H_r^1}{2} \right)^2 \right) \cdot \forall j \geq 3. \]

From

\[ \tilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - g(h(e_2, e_2), h(e_1, e_1)) + g(h(e_2, e_1), h(e_2, e_1)), \]

we get

\[ \tilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (n + 1) \left( \sum_{r=1}^{n} \left( \frac{H_r^1}{2} \right)^2 \right) + \left( \sum_{r=1}^{n} \left( \frac{H_r^1}{2} \right)^2 \right). \]

By combining (3.13) and (3.14), we get

\[ \text{Ric}(e_2) - \frac{(n - 1)c}{4} = (n + 1) \sum_{r=1}^{3} \left( \frac{H_r^1}{2} \right)^2 - \left( \sum_{r=1}^{3} \left( \frac{H_r^1}{2} \right)^2 \right) + (n - 2) \sum_{r=1}^{3} \left( \frac{H_r^1}{2} \right)^2 = 2(n - 1) \sum_{r=1}^{3} \left( \frac{H_r^1}{2} \right)^2. \]

On the other hand from the equality assumption, we have

\[ \text{Ric}(e_2) - \frac{(n - 1)c}{4} = \frac{n(n - 1)c}{4} = n(n - 1) \sum_{r=1}^{3} \left( \frac{H_r^1}{2} \right)^2. \]

Therefore, we have

\[ n(n - 1) \sum_{r=1}^{3} \left( \frac{H_r^1}{2} \right)^2 = 2(n - 1) \sum_{r=1}^{3} \left( \frac{H_r^1}{2} \right)^2. \]
Since \( n \geq 3 \), we have \( H_1^3 = H_2^3 = H_3^3 = 0 \). Therefore, \( (h^{\phi_r}_{jk}) \) are all zero \((r = 1, 2, 3)\) and \( x \) is a totally geodesic point.

Case (2) \( n = 2 \). From (3.12) we have
\[
\begin{align*}
  h^{\phi_r}_{11} &= 3h^{\phi_r}_{22}, \\
  h^{\phi_r}_{22} &= 3h^{\phi_r}_{11}, \\
  r &= 1, 2, 3.
\end{align*}
\]
Since \( X \) can be any unit vector, we may assume that the mean curvature vector is in \( Q(X) \). Then the second fundamental form takes the following form:
\[
\begin{align*}
  h(e_1, e_1) &= 3\mu_1 I(e_1) + 3\mu_2 J(e_1) + 3\mu_3 K(e_1), \\
  h(e_2, e_2) &= \mu_1 I(e_2) + \mu_2 J(e_1) + \mu_3 K(e_1), \\
  h(e_1, e_2) &= \mu_1 I(e_2) + \mu_2 J(e_2) + \mu_3 K(e_2),
\end{align*}
\]
for some functions \( \mu_1, \mu_2 \) and \( \mu_3 \) with respect to some local orthonormal frame field. It follows from (2.4) that \( x \) is an H-umbilical point with \( \lambda_r = 3\mu_r, r = 1, 2, 3 \).

The converse can be proved by simple computation.

\[\Box\]

**Remark 3.1.** Theorem 3.1 is an improvement of a result in [1, page 38] for Lagrangian submanifolds. Theorem 3.1 is also an extension of a theorem in [5] for Lagrangian submanifolds in quaternion space forms.

**Remark 3.2.** In quaternion space forms, Theorem 3.1 is an improvement of Corollary 2.1 in [8] for Lagrangian submanifolds.

From Theorem 3.1, we have the following

**Corollary 3.2.** Let \( M \) be a Lagrangian submanifold of real dimension \( n (n \geq 2) \) in a \( 4n \)-dimensional quaternion space form \( \tilde{M}^n(c) \). If
\[
\text{Ric}(X) = \frac{n-1}{4}(c + n\|H\|^2)
\]
for any unit tangent vector \( X \) of \( M \), then either \( M \) is a totally geodesic submanifold in \( \tilde{M}^n(c) \) or \( n = 2 \) and \( M \) is a Lagrangian H-umbilical submanifold of \( \tilde{M}^n(c) \) with \( \lambda_r = 3\mu_r, r = 1, 2, 3 \).

**Remark 3.3.** Corollary 3.2 is an improvement of Theorem 3.1 in [7] for Lagrangian submanifolds in quaternion space forms.

**Remark 3.4.** Theorem 3.1 and Corollary 3.2 give a complete solution to Problem 4.6 in [12].

**References**


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