1.6 Problem #7

Alka Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$).

When it plops and fizzes, it produces sodium citrate, water, and carbon dioxide,

$Na_3C_6H_5O_7,\ H_2O,\ CO_2$.

We will balance the reaction equation

$NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2$. using a system of equations.

Here the unknowns are the number of each types of molecule in the reaction equation:

$$x_1\ NaHCO_3 + x_2\ H_3C_6H_5O_7 \rightarrow x_3\ Na_3C_6H_5O_7 + x_4\ H_2O + x_5\ CO_2.$$

To get the linear equation, we use the fact that matter is neither created or destroyed in chemical reactions, or more bluntly that the number of atoms of each element is the same before and after the reaction. Proceeding in alphabetical order, we have:

C $\quad x_1 + 6x_2 = 6x_3 + x_5$

H $\quad x_1 + 8x_2 = 5x_3 + 2x_4$

Na $\quad x_1 = 3x_3$

O $\quad 3x_1 + 7x_2 = 7x_3 + x_4 + 2x_5$

We rewrite these equations:

$$x_1 + 6x_2 - 6x_3 - x_5 = 0$$

$$x_1 + 8x_2 - 5x_3 - 2x_4 = 0$$

$$x_1 - 3x_3 = 0$$

$$3x_1 + 7x_2 - 7x_3 - x_4 - 2x_5 = 0$$

We consider the associated augmented matrix:

$$\begin{bmatrix} 1 & 6 & -6 & 0 & -1 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{bmatrix}$$

After a few hours of sweat and toil we have calculated the reduced form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -0 & -1/3 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Translating the augmented matrix back into a system of equations, we have:

$$\begin{aligned} x_1 &= x_5 \\ 3x_2 &= x_5 \\ 3x_3 &= x_5 \\ x_4 &= x_5 \\ x_5 &\text{ is free} \end{aligned}$$

Recall that the reaction equation is:

$$x_1\ NaHCO_3 + x_2\ H_3C_6H_5O_7 \rightarrow x_3\ Na_3C_6H_5O_7 + x_4\ H_2O + x_5\ CO_2$$

The variables represent numbers of molecules, so they need to be whole numbers.

The particular solution using the smallest possible numbers is

(3,1,1,3,3),

giving $3\ NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + 3\ H_2O + 3\ CO_2$.