MATH 1112 – Exam 2 (Version 2) Solutions

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Instructions. Your work on this exam will be graded according to two criteria: **mathematical correctness** and **clarity of presentation**. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using **complete sentences** where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an "answer" with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator on this exam but you may not use any books or notes.

1. Use the identity

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

and the even/odd and cofunction identities to prove the all three of the identities

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

In order to receive full credit for this problem, you must use correct notation and include a sufficient amount of detail in your proofs.

2. Use an appropriate sum or difference identity to find the exact value $\sin(15^\circ)$. (A calculator answer will not suffice.)

Solution: Note that $15^\circ = 45^\circ - 30^\circ$. Thus

$$\begin{aligned}\sin(15°) &= \sin(45° - 30°)\\ &= \sin(45°)\cos(30°) - \cos(45°)\sin(30°)\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

3. Prove the identity

$$\frac{1 + \sin(2x) + \cos(2x)}{1 + \sin(2x) - \cos(2x)} = \cot(x).$$

Proof:

$$\begin{aligned}\frac{1 + \sin(2x) + \cos(2x)}{1 + \sin(2x) - \cos(2x)} &= \frac{1 + 2\sin(x)\cos(x) + \cos^2(x) - \sin^2(x)}{1 + 2\sin(x)\cos(x) - \cos^2(x) + \sin^2(x)}\\ &= \frac{1 - \sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x)}{1 - \cos^2(x) + 2\sin(x)\cos(x) + \sin^2(x)}\\ &= \frac{\cos^2(x) + 2\sin(x)\cos(x) + \cos^2(x)}{\sin^2(x) + 2\sin(x)\cos(x) + \sin^2(x)}\\ &= \frac{2\cos^2(x) + 2\sin(x)\cos(x)}{2\sin^2(x) + 2\sin(x)\cos(x)}\\ &= \frac{2\cos(x)(\cos(x) + \sin(x))}{2\sin(x)(\sin(x) + \cos(x))}\\ &= \frac{\cos(x)}{\sin(x)}\\ &= \cot(x)\end{aligned}$$

Q.E.D.

4. Decide whether each of the following statements is true or false.

a. If K is any real number such that $-1 \le K \le 1$, then $0 \le \sin^{-1}(K) \le \pi$. (True / [False])

b. If K is any real number such that $-1 \le K \le 1$, then $0 \le \cos^{-1}(K) \le \pi$. ([True] / False)

c. If K is any real number such that $K > 1$, then $\tan^{-1}(K)$ is not defined. (True / [False])

d. If θ is any real number such that $\tan(\theta) = K$, then $\tan^{-1}(K) = \theta$. (True / [False])

e. If θ is any real number such that $0 \le \theta < \pi$ and $\sin(\theta) = K$, then $\sin^{-1}(K) = \theta$. (True / [False])

5. Find all solutions of the equation

$$\cos(x) = -\frac{1}{2}$$

Solution. By our knowledge of the unit circle, we find that the solutions are

$$x = \frac{2\pi}{3} + n \cdot 2\pi$$

and

$$x = \frac{4\pi}{3} + n \cdot 2\pi$$

where n can be any integer.

6. Find all solutions of the equation

$$2\sin^2(x) - \sqrt{3}\,\sin(x) = 0$$

that lie in the interval $[0, 2\pi]$.

Solution: We can write the equation to be solved as

$$\sin(x)\left(2\sin(x) - \sqrt{3}\right) = 0.$$

Thus solutions must either satisfy

$$\sin(x) = 0$$

or

$$2\sin(x) - \sqrt{3} = 0$$

which can also be written as

$$\sin(x) = \frac{\sqrt{3}}{2}.$$

By our knowledge of the unit circle we know that the solutions of $\sin(x) = 0$ that lie in the interval $[0, 2\pi]$ are $x = 0$, $x = \pi$ and $x = 2\pi$ and we know that the solutions of $\sin(x) = \sqrt{3}/2$ that lie in the interval $[0, 2\pi]$ are $x = \pi/3$ and $x = 2\pi/3$. Thus the solutions of the given equation that lie in the interval $[0, 2\pi]$ are $x = 0,\ \pi/3, 2\pi/3, \pi$ and 2π.