Structures and repeated reasoning: Keys to decimal number teaching and learning

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Tad Watanabe
Kennesaw State University
twatanab@kennesaw.edu
http://ksuweb.kennesaw.edu/~twatanab/

KENNESAW STATE UNIVERSITY®
College of Science and Mathematics
Department of Mathematics

Fractions and Decimal Numbers

Grade	Fractions	Decimal Numbers
1 & 2	Foundations – partitioning of shapes (1.G & 2.G)	
3	Formal introduction – focus on unit fractions (3.NF)	
4	Equivalent fractions +/−: like denominators ×: by whole numbers (4.NF)	Decimal numbers as "decimal notation" of fractions – 10th and 100th (4.NF)
5	×: by fractions ÷: whole number ÷ unit fraction unit fraction ÷ whole number (5.NF)	Decimal numbers through 1000th (5.NBT) +/−/×/÷: through 100th With concrete models, drawings, strategies based on place value, properties of operations; relate strategy to written method (5.NBT)
6	÷: fraction ÷ fraction Invert-and-multiply algorithm (6.NS)	Fluency with the standard algorithms (6.NS)

Fractions and Decimal Numbers

- Decimal numbers have the characteristics of both fractions and whole numbers.
  - Decimal fractions: fractions with denominators of powers of 10
  - Extending decimal numeration system.

- Teaching and learning of decimal numbers should take advantage of characteristics familiar to students.

Fractions and Decimal Numbers

- Fractions and decimal numbers are two notation systems for numbers that require units less than one.
- Numbers are expressed in terms of units.
  - Units for decimal numbers: powers of 10
  - Units for fractions: unit fractions
    - $\frac{1}{D}$ = one of D equal partitioning of 1

Structure of Decimal Numeration

- Positional: where a numeral is written matters.

Positional and Non-positional System

5	五
55	五十五
505	五百五
5005	五千五
50005	五万五

Positional and Non-positional System

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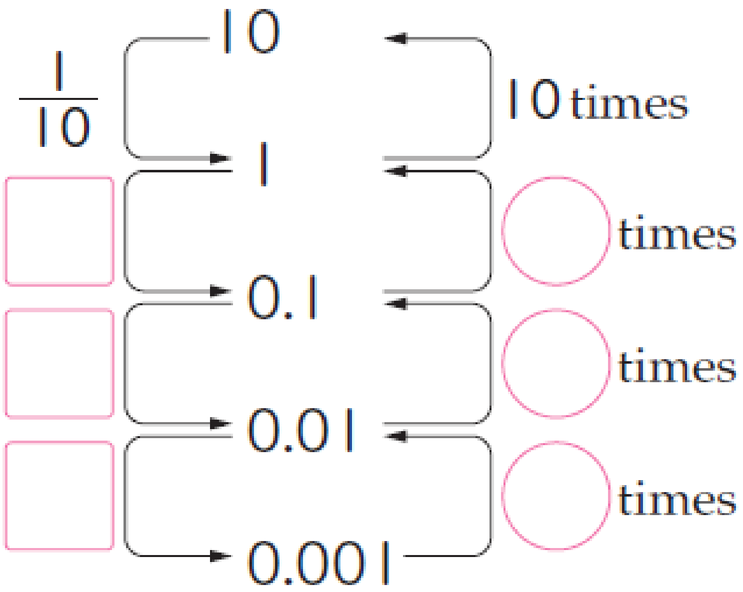
50005	五万五

Structure of Decimal Numeration

- Positional: where a numeral is written matters.
- Each position (place) represents a specific value.
- Adjacent positions (places) are always in 1 to 10 relationship – 10 of a smaller units make up 1 of the next larger unit.

3 Write the number that goes in each of the ☐ on the right.
Also, write a number in each of the ◯ .



Tokyo Shoseki (2010) Gr.4 p. A94

Multiplication/division by 10

3 If you multiply 25 10 times, then another 10 times, what number will you get?

1 How many times as much will it be if you multiply a number 10 times and then another 10 times?

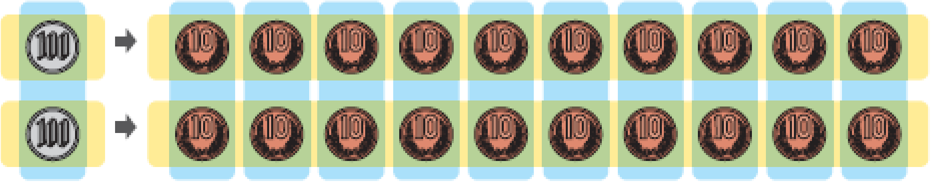
25 —10 times→ 250 —10 times→ 2500

25 → 2500: ☐ times

	One thousands	Hundreds	Tens	Ones
			2	5
25 × 10…		2	5	0
250 × 10…	2	5	0	0

10 times, 10 times, 100 times

4 What number is 200 divided by 10?



200 —Divided by 10→ 20

200 ÷ 10 = ☐

Hundreds	Tens	Ones
2	0	0
	2	0

Dividec by 10

Tokyo Shoseki (2010) Gr.3 p. A86

Multiplication/division by 10

2 Investigate what happens to 3.75 when it is made 10 times and 100 times as much.

Hundreds place	Tens place	Ones place	Tenths place	Hundredths place	Thousandths place
		3.	7	5	

(Arrows beside the rows: 10 times, 10 times; 100 times overall)

3 Investigate what happens to 25.7 when it is made $\frac{1}{10}$ and $\frac{1}{100}$ as much.

Hundreds place	Tens place	Ones place	Tenths place	Hundredths place	Thousandths place
	2	5.	7		

(Arrows beside the rows: $\frac{1}{10}$, $\frac{1}{10}$; $\frac{1}{100}$ overall)

Tokyo Shoseki (2010) Gr.5 pp. A8 & A9

Contrasting fractions and decimal numbers

Math Story

The Differences Between the Structure of Whole / Decimal Numbers and Fractions

The place values for whole numbers and decimal numbers are always 10 times or $\frac{1}{10}$ of the adjacent place values.

10 times 10 times 10 times 10 times 10 times

Hundreds place	Tens place	Ones place	Tenths place	Hundredths place	Thousandths place

$\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$

"The place value is $\frac{1}{10}$ of the adjacent place" means that the new place value is one of 10 equally divided parts of the original place value, doesn't it?

Takumi

On the other hand, fractions are created to express amounts that are equally divided. Therefore, unlike whole numbers and decimal numbers the number of equal parts in a fraction does not have to be just 10.

Fractions were already being used in Egypt and Babylon (the southern part of today's Iraq) more than 4000 years ago, much earlier than decimal

Fractions were already being used in Egypt and Babylon (the southern part of today's Iraq) more than 4000 years ago, much earlier than decimal numbers. In Egypt, fractions with 1 as the numerator, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, ... were used to express one of 2, 3, 4, 5, ... equal parts.

In Babylon, they used the denominator of 60 and different numbers for the numerator. 60 was useful because it can be divided evenly by many whole numbers, and divided into equal parts in many different ways.

So we can see that the idea of dividing evenly is used differently in whole numbers, decimal numbers, and in fractions.

Tokyo Shoseki (2010) Gr.5 p. A7

Relative Size of Numbers

- With the decimal numeration system, a number is represented as accumulation of units (powers of 10).
- 23.45 is made of
  - 2 units of 10
  - 3 units of 1
  - 4 units of 0.1
  - 5 units of 0.01

Relative sizes of 0.67

- 6 units of 0.1, and
- 7 units of 0.01

  Or

  - 67 units of 0.01

    Or

    - 670 units of 0.001

      Or

      - 6700 units of 0.0001

        Or …

Relative sizes of

- $\frac{2}{3}$ is made of 2 units of $\frac{1}{3}$

  or

  - $\frac{2}{3}$ is made of 4 units of $\frac{1}{6}$

  or

    - $\frac{2}{3}$ is made of 6 units of $\frac{1}{9}$

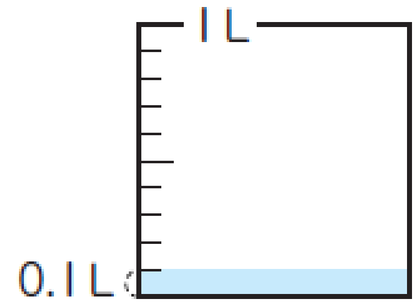
    or …

Structure of Decimal Numeration

- Positional: where a numeral is written matters.
- Each position (place) represents a specific value.
- Adjacent positions (places) are always in 1 to 10 relationship – 10 of a smaller units make up 1 of the next larger unit.
- Each position must have one and only one numeral.
  - 0 as a place holder
  - Exceptions: leading 0's for whole numbers and trailing 0's for decimal numbers

One and only one numeral in each place

3 What is the amount of water if there are nine 0.1 L?
Also, what is the amount of water if there are ten 0.1 L?



Tokyo Shoseki (2010) Gr.3 p. B15

2 What is the length of six $\frac{1}{5}$ m, seven $\frac{1}{5}$ m, and so on?

? Let's think about how to express fractions greater than 1.

The length of six $\frac{1}{5}$ m is expressed as $\frac{6}{5}$ m.

Gr.3 p. B49

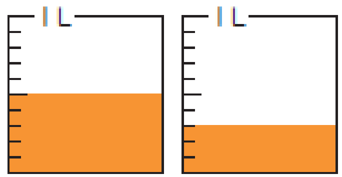
Regularity in Repeated Reasoning

- Addition and subtraction of decimal numbers
- Thinking in terms of units other than 1

3 Addition and Subtraction of Decimal Numbers

1 A big bottle contains 0.5L of juice and a small bottle contains 0.3L of juice.
How much juice is there altogether?

Math Sentence ____________________



? **Let's think about how to calculate.**

★1 How many 0.1L are in 0.5L and 0.3L each?

Hiroki

If we think 0.1L as a unit, □+□ so…

★2 Explain how the calculation on the right was done.

$0.8+0.2=1$

2 Think about how to calculate 0.4+0.7.

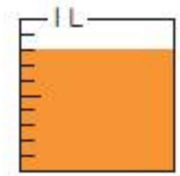
★3 If you think of 0.1 as a unit, what kind of calculation does 0.4+0.7 become?

Yumi

0.4 is made of □ 0.1's, and 0.7 is made of □ 0.1's, so …

3 There is 0.8L of juice.
She drank 0.3L of it.
How many L of juice are left?

Math Sentence ____________________



? **Let's think about how to calculate.**

★1 How many 0.1L are in 0.8L and 0.3L each?

Miho

If we think 0.1L as a unit, □−□ so…

★2 Explain how the calculation on the right was done.

$1-0.4=0.6$

4 Think about how to calculate 1.4−0.6.

★3 If you think of 0.1 as a unit, what kind of calculation does 1.4−0.6 become?

Takumi

1.4 is made of □ 0.1's, and 0.6 is made of □ 0.1's, so…

Tokyo Shoseki (2010) Gr.3 pp. B19 & B20

1

There is $\frac{3}{10}$ L of juice in a carton and $\frac{2}{10}$ L in a bottle. How much juice is the altogether in L?

1 Write a math sentence.

Math Sentence ______________________

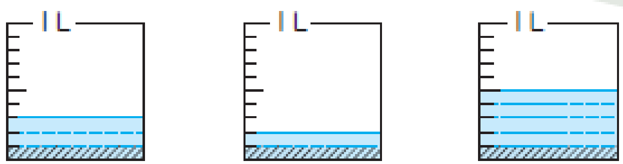
? Let's investigate to find out if we can do addition with fractions.

If they are decimal numbers, 0.3 + 0.2 = 0.5, but…

Kaori

2 Let's think about how to calculate $\frac{3}{10}+\frac{2}{10}$.

Think about it by looking at the fractions as how many $\frac{1}{10}$ L.



☐ of $\frac{1}{10}$ L   ☐ of $\frac{1}{10}$ L   ☐ of $\frac{1}{10}$ L

$\frac{3}{10}$ + $\frac{2}{10}$ = ☐

2

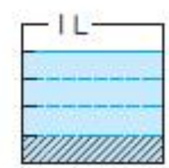
There is $\frac{4}{5}$ L of juice.
If a girl drinks $\frac{1}{5}$ L of juice, how much juice will be left in L?

4 Write a math sentence.

Math Sentence ______________________

? Let's investigate to find out if we can do subtraction with fractions.

5 Let's think about how to calculate $\frac{4}{5}-\frac{1}{5}$.

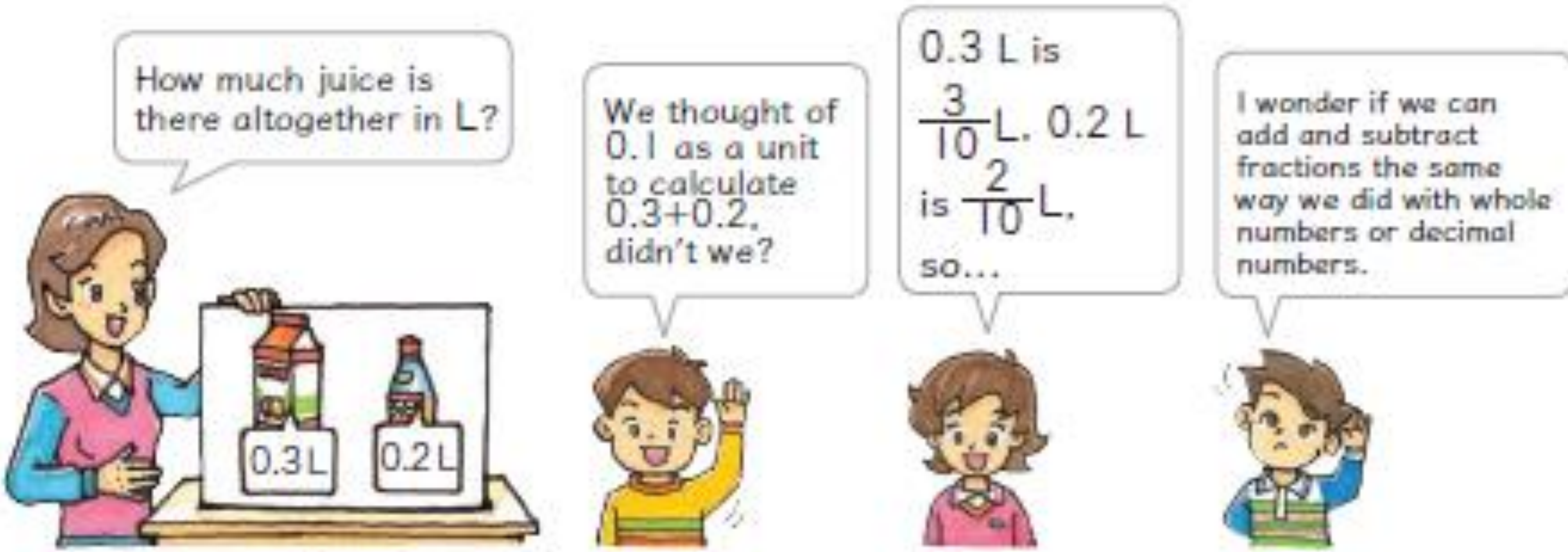


$\frac{4}{5}-\frac{1}{5}$ = ☐

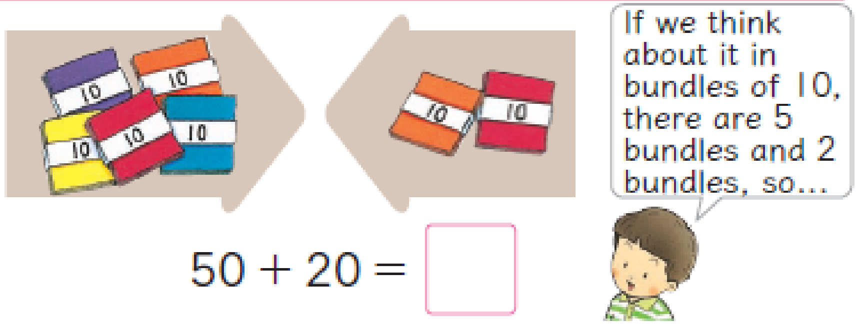
If you think about it with $\frac{1}{5}$ L as a unit…

Tokyo Shoseki (2010) Gr.3 pp. B51 & B52

Emphasizing repeated reasoning

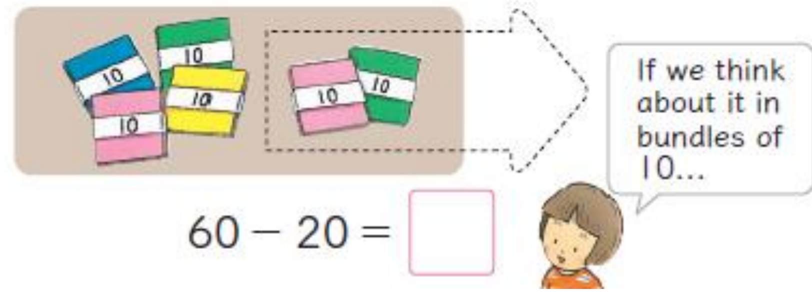


1 How many pieces of colored paper are there altogether?



$50 + 20 =$ □

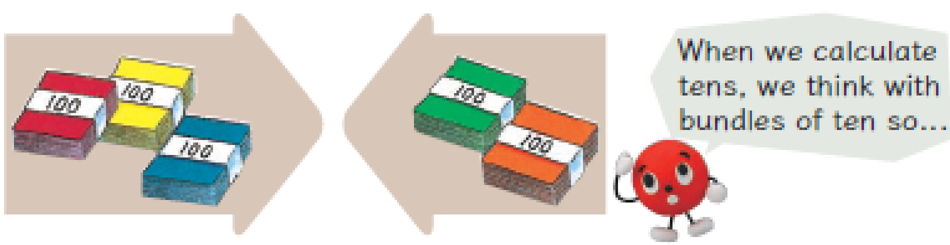
3 There are 60 pieces of colored paper.
If we use 20 pieces, how many will be left?



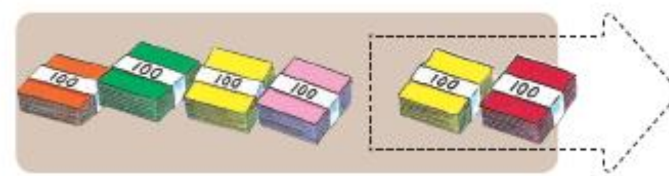
$60 - 20 =$ □

Tokyo Shoseki (2010) Gr.1 p. 128

3 Think about how to calculate 300+200.



4 Think about how to calculate 600−200.



2 Explain the calculation by thinking the same way as 3.

$600 - 200 =$ □

Tokyo Shoseki (2010) Gr.2 pp. A60 & A61

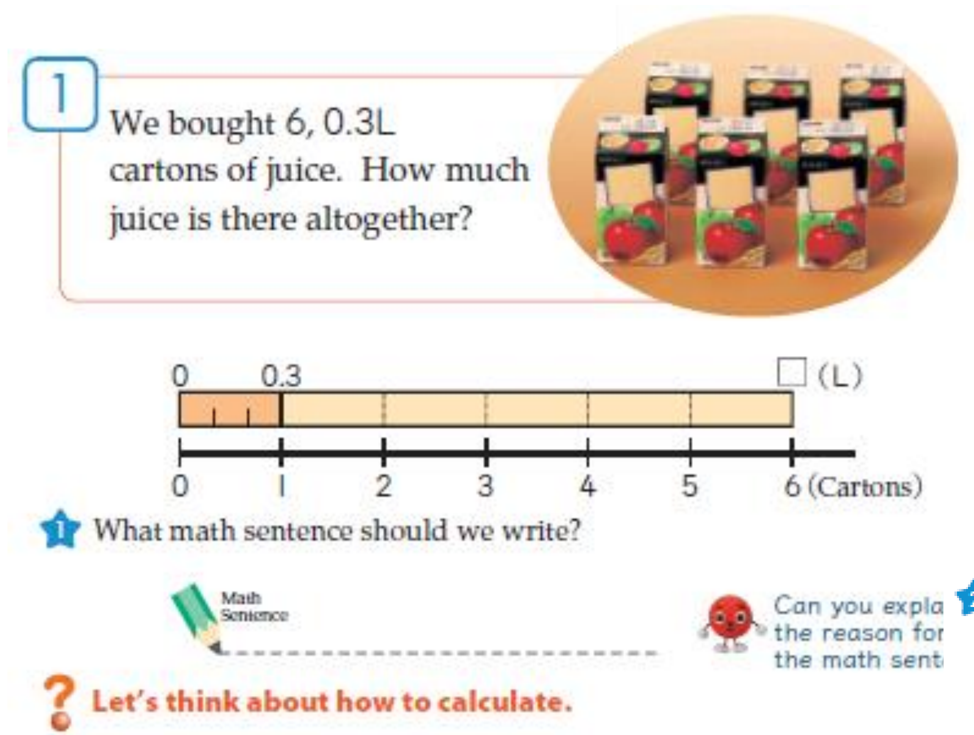
Multiplication of Decimal Numbers

- Multiplying decimal numbers by whole numbers in Grade 4
- Multiplying by decimal numbers in Grade 5

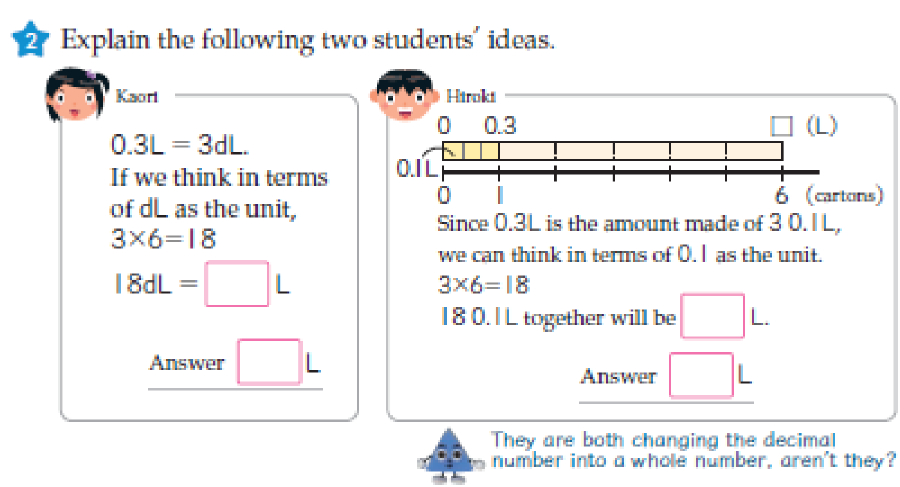
- CCSS 4.NF.4: multiplying fractions by whole numbers
- CCSS 5.NF.4: multiplying by fractions

Multiplying decimal numbers by whole numbers

- Continue to make use of decimal units



Tokyo Shoseki (2010) Gr.4 pp. B73 & B74



Multiplying decimal numbers by whole numbers

- Make use of property of multiplication

Think about how to calculate (decimal number)×(whole number) based on the way we calculate (whole number)×(whole number).

$0.3 \times 6 = 1.8$

↓ □ times ↓ □ times $\frac{1}{10}$ (Divide by 10)

$3 \times 6 = 18$

The product of 0.3×6 can be calculated by first making 0.3 10 times as much, then by calculating 3×6, and then by dividing the product by 10.

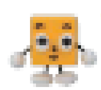
Tokyo Shoseki (2010) Gr.4 p. B74

The answer for 5×30 is the same as 10 times as much as 5×3. Therefore, the answer is the same as placing a 0 to the right of 15.

$5 \times 3 = 15$

↓ 10 times ↓ 10 times

$5 \times 30 = 150$



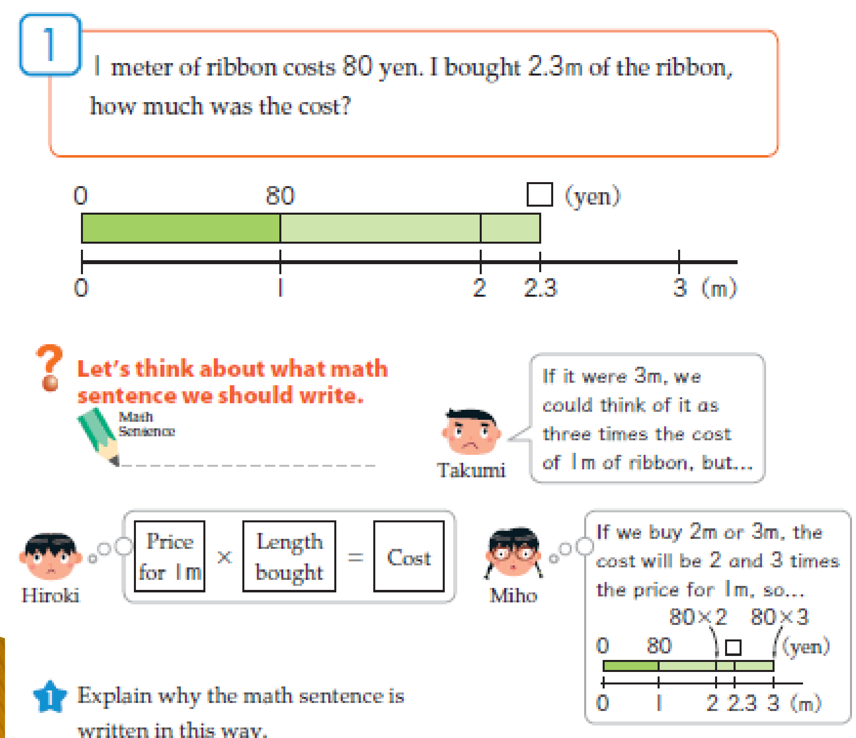
When the number in the multiplier becomes 10 times as much, the answer also becomes 10 times as much.

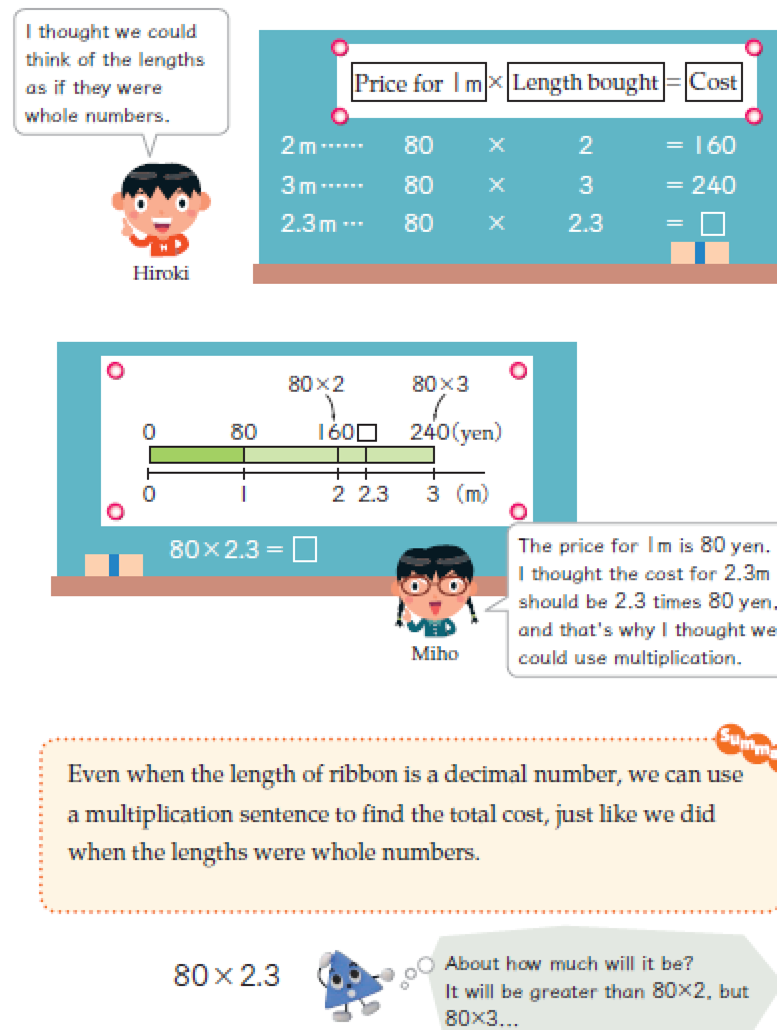
Tokyo Shoseki (2010) Gr.3 p. B64

Multiplying by decimal numbers

- Making sense of multiplication by decimal numbers first

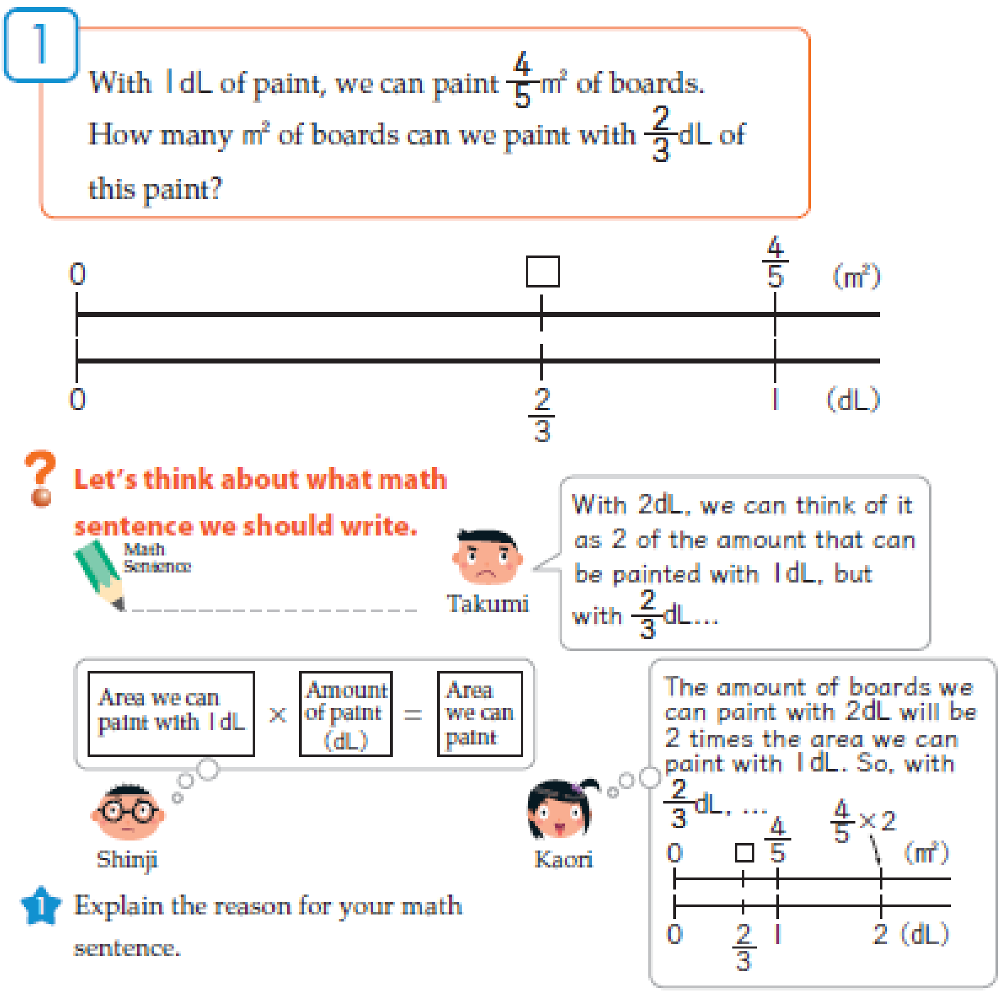
Tokyo Shoseki (2010) Gr.5 pp. A31 & A32





Multiplying by fractions

- Making sense of multiplication by fractions first



1

With 1 dL of paint, we can paint $\frac{4}{5}$ m² of boards. How many m² of boards can we paint with $\frac{2}{3}$ dL of this paint?

0 □ $\frac{4}{5}$ (m²)

0 $\frac{2}{3}$ 1 (dL)

Let's think about what math sentence we should write.

Math Sentence

Takumi: With 2dL, we can think of it as 2 of the amount that can be painted with 1dL, but with $\frac{2}{3}$ dL…

Area we can paint with 1 dL × Amount of paint (dL) = Area we can paint

Shinji

Kaori: The amount of boards we can paint with 2dL will be 2 times the area we can paint with 1dL. So, with $\frac{2}{3}$ dL, …

0 □ $\frac{4}{5}$ $\frac{4}{5}\times 2$ (m²)

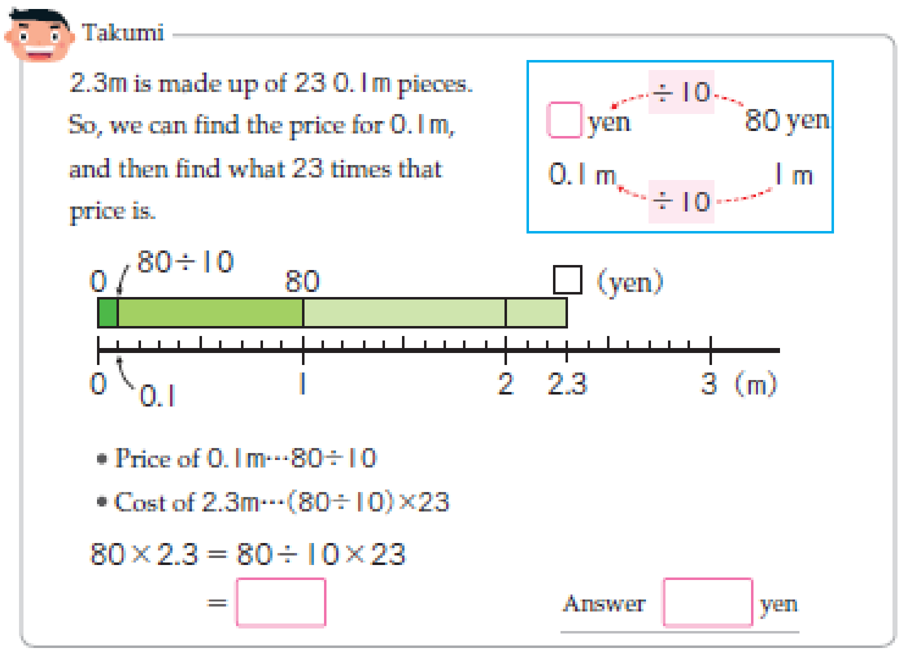
0 $\frac{2}{3}$ 1 2 (dL)

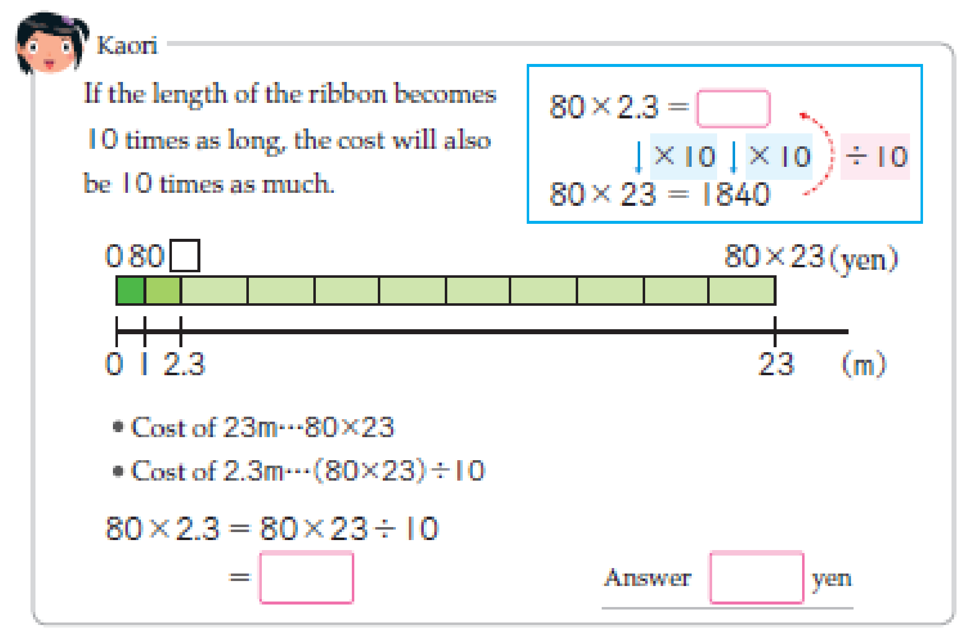
1 Explain the reason for your math sentence.

Tokyo Shoseki (2010) Gr.6 p. A34

Ways to multiply by decimal numbers

- Thinking in terms of decimal units
- Using property of multiplication

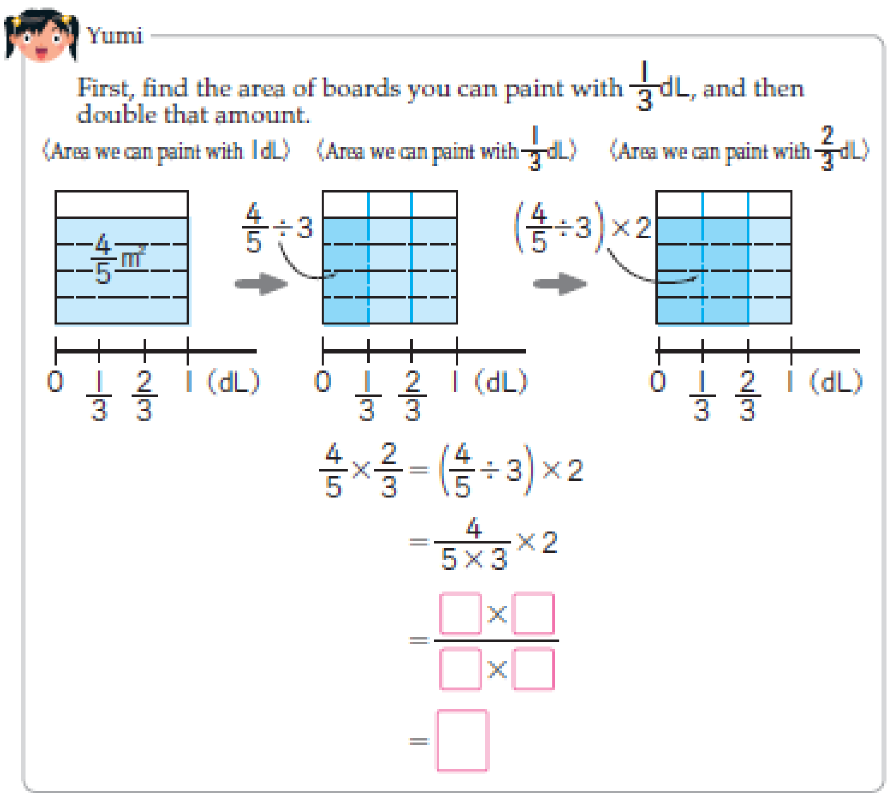


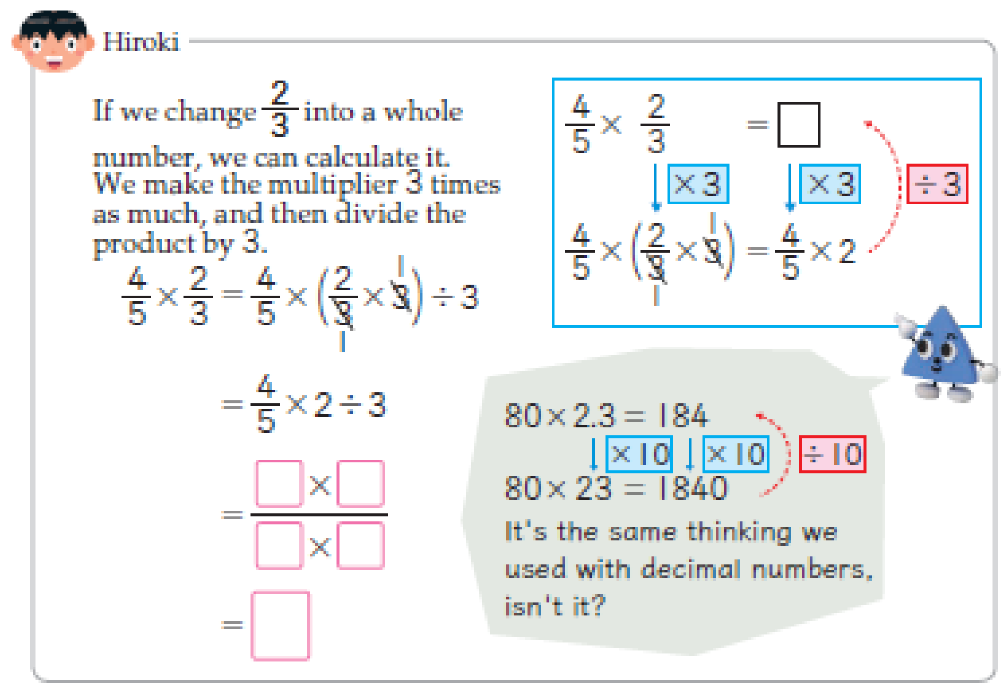


Tokyo Shoseki (2010) Gr.5 p. A33

Ways to multiply by fractions

- Thinking in terms of fraction units
- Using property of multiplication





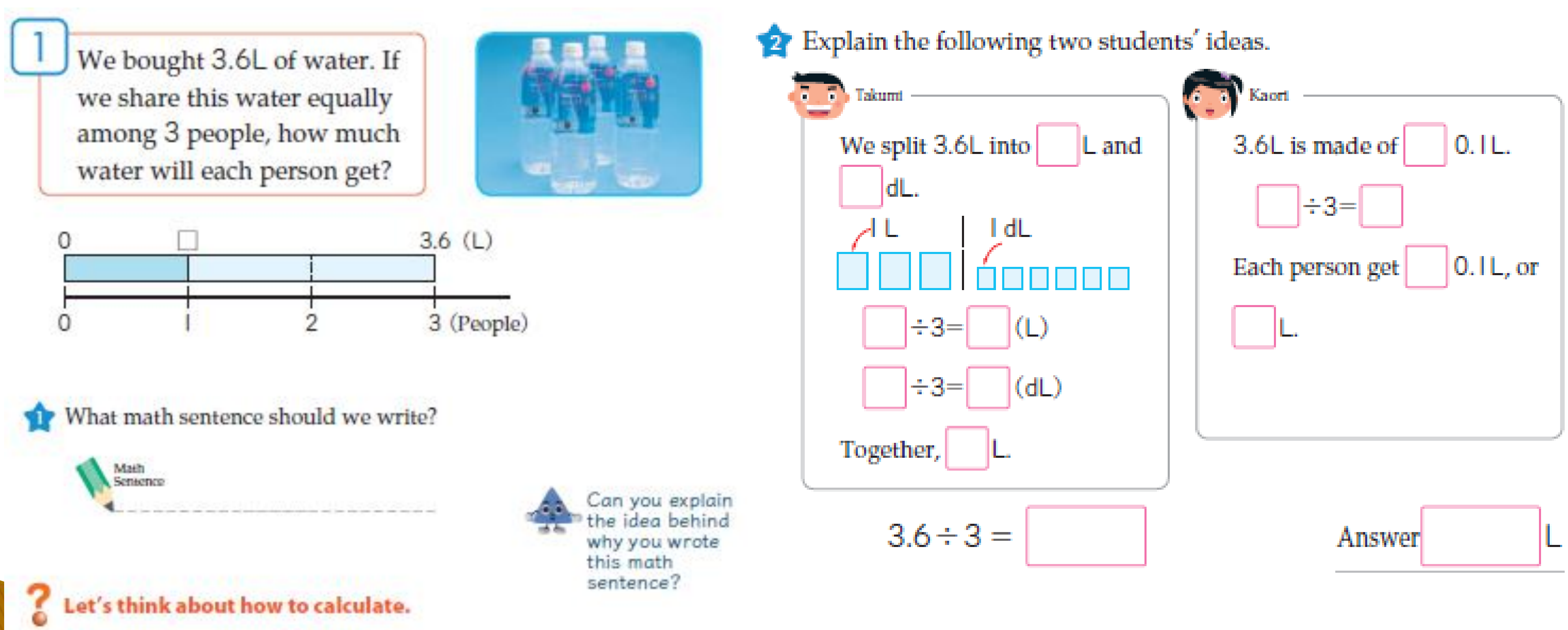
Tokyo Shoseki (2010) Gr.6 p. A25

Division of decimal numbers

- Dividing decimal numbers by whole numbers in Grade 4
- Dividing by decimal numbers in Grade 5

Dividing decimal numbers by whole numbers

- Making use of context
- Making use of decimal units



1 We bought 3.6L of water. If we share this water equally among 3 people, how much water will each person get?

0 □ 3.6 (L)

0 1 2 3 (People)

1 What math sentence should we write?

Math Sentence

Can you explain the idea behind why you wrote this math sentence?

? Let's think about how to calculate.

2 Explain the following two students' ideas.

Takumi

We split 3.6L into □ L and □ dL.

1 L | 1 dL

□ ÷3= □ (L)

□ ÷3= □ (dL)

Together, □ L.

Kaori

3.6L is made of □ 0.1L.

□ ÷3= □

Each person get □ 0.1L, or □ L.

3.6 ÷ 3 = □

Answer □ L

Tokyo Shoseki (2010) Gr.4 pp. B80 & B81

Dividing decimal numbers by whole numbers

- What if the dividend is not evenly divisible?
  - Remainder
  - Dividing on

7 Calculate 46.7÷3 using the division algorithm. Calculate the quotient to the ones place, and find the remainder.

? **Let's think about the size of the remainder when we divide decimal numbers.**

8 If we share 6L of juice equally among 4 people, how much juice will each person get?

Math Sentence

Takumi

6÷4=1 remainder 2, but …

6L is made of 60 0.1L, so, …

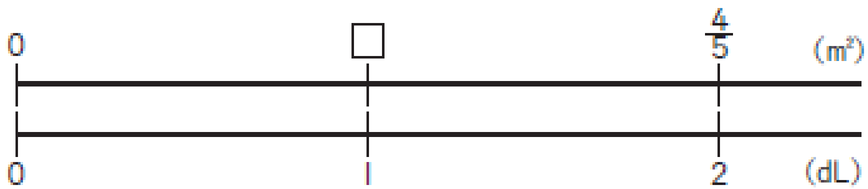
Kaori

? **Let's think about how to continue dividing with the division algorithm.**

Tokyo Shoseki (2010) Gr.4 pp. B86 & B87

Dividing fractions by whole numbers

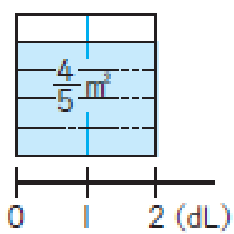
3 You can paint $\frac{4}{5}$ m² of boards with 2dL of paint.
How many m² can you paint with 1 dL of this paint?



What math sentence do we need to write?

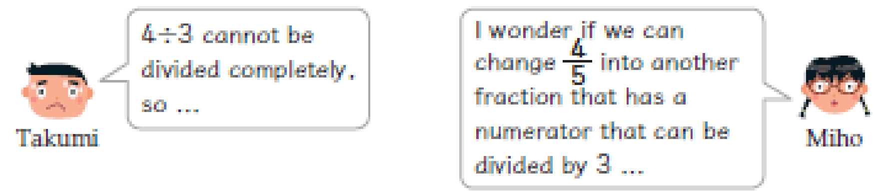
Math Sentence

Can you explain your answer?



Let's think about how to calculate.

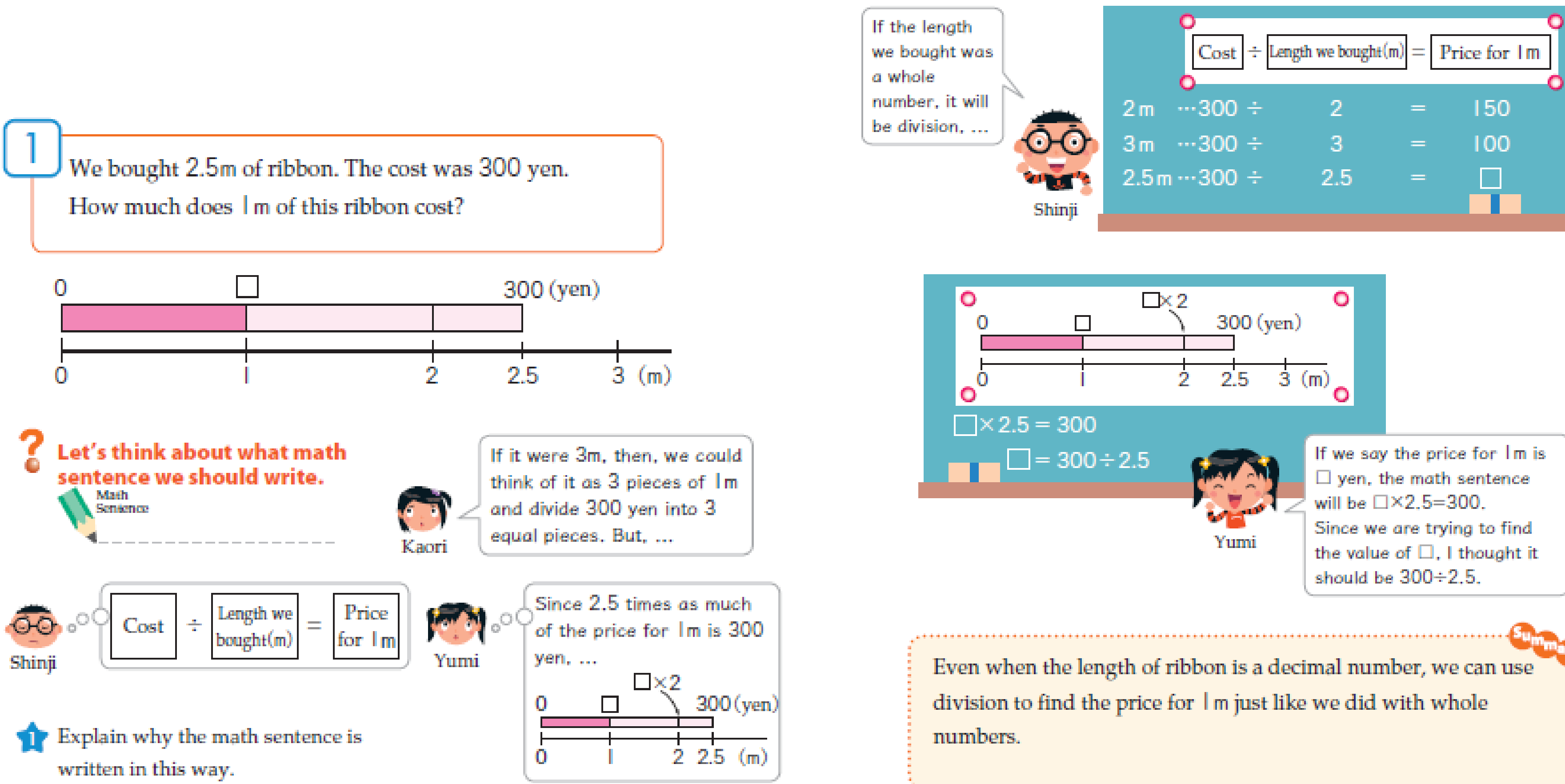
4 Think about how to calculate $\frac{4}{5} \div 3$.



Tokyo Shoseki (2010) Gr.5 pp. B91 & B92

Dividing by decimal numbers

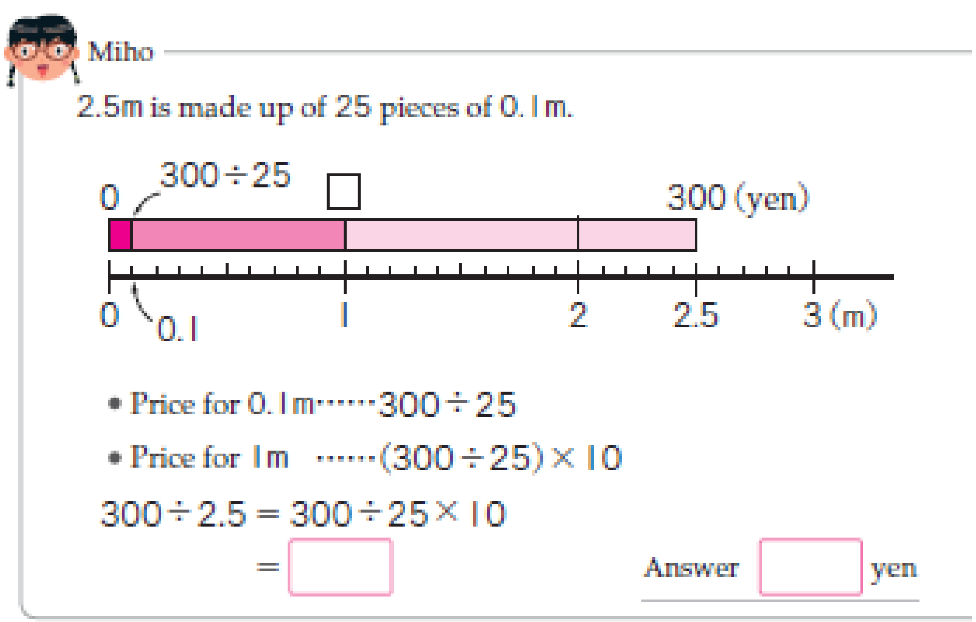
- Making sense of multiplication by decimal numbers first

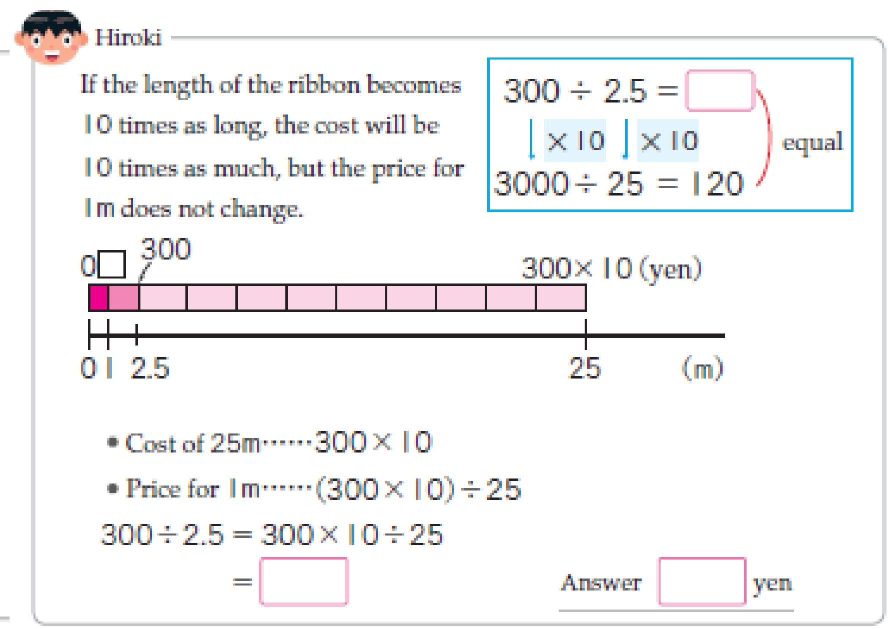


Tokyo Shoseki (2010) Gr.5 pp. A45 & A46

Ways to divide by decimal numbers

- Thinking in terms of decimal units
- Using property of division

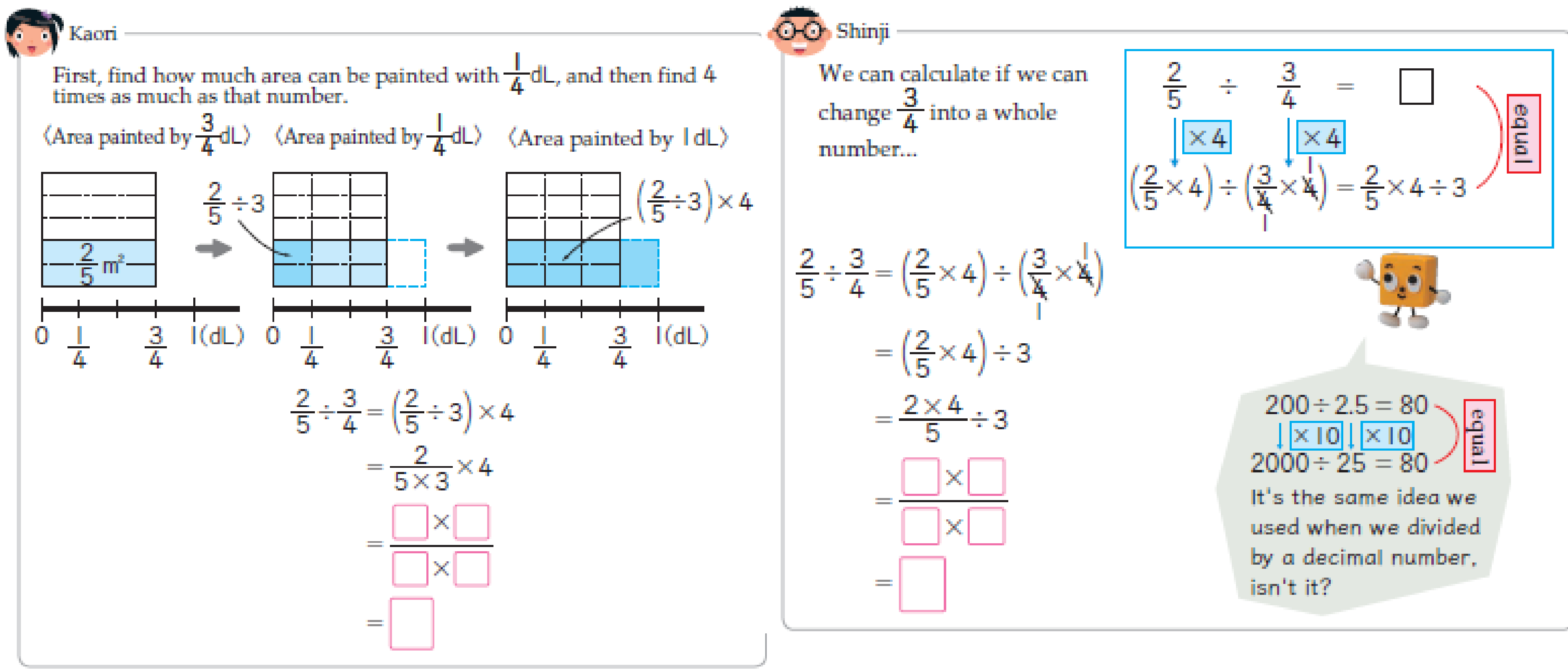




Tokyo Shoseki (2010) Gr.5 p. A47

Ways to divide by fractions

- Thinking in terms of fraction units
- Using property of division



Tokyo Shoseki (2010) Gr.5 p. A47

Final Thoughts

- In order for students to look for and make use of structures in learning of decimal numbers, structures must become a focus in their learning of whole numbers (and fractions).
- It is helpful to have a curriculum flow that makes use of structures as a theme.

Final Thoughts

- In order for students to look for and express regularity in repeated reasoning with decimal numbers, reasoning must become a focus in mathematics lessons.
- Tasks for lessons must be carefully chosen so that desired reasoning is more likely to arise from students.

Thank you!