

Structures and repeated reasoning: Keys to decimal number teaching and learning

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Fractions and Decimal Numbers

Grade	Fractions	Decimal Numbers
1 & 2	Foundations – partitioning of shapes (1.G & 2.G)	
3	Formal introduction – focus on unit fractions (3.NF)	
4	Equivalent fractions + / –: like denominators ×: by whole numbers (4.NF)	Decimal numbers as “decimal notation” of fractions – 10 th and 100 th (4.NF)
5	×: by fractions ÷: whole number ÷ unit fraction unit fraction ÷ whole number (5.NF)	Decimal numbers through 1000 th (5.NBT) + / – / × / ÷: through 100 th With concrete models, drawings, strategies based on place value, properties of operations; relate strategy to written method (5.NBT)
6	÷: fraction ÷ fraction Invert-and-multiply algorithm (6.NS)	Fluency with the standard algorithms (6.NS)

Fractions and Decimal Numbers

- ▶ Decimal numbers have the characteristics of both fractions and whole numbers.
 - Decimal fractions: fractions with denominators of powers of 10
 - Extending decimal numeration system.
- ▶ Teaching and learning of decimal numbers should take advantage of characteristics familiar to students.

Fractions and Decimal Numbers

- ▶ Fractions and decimal numbers are two notation systems for numbers that require units less than one.
- ▶ Numbers are expressed in terms of units.
 - Units for decimal numbers: powers of 10
 - Units for fractions: unit fractions
 - $\frac{1}{D}$ = one of D equal partitioning of 1

Structure of Decimal Numeration

- ▶ Positional: where a numeral is written matters.


Positional and Non-positional System

5	五
55	五十五
505	五百五
5005	五千五
50005	五万五

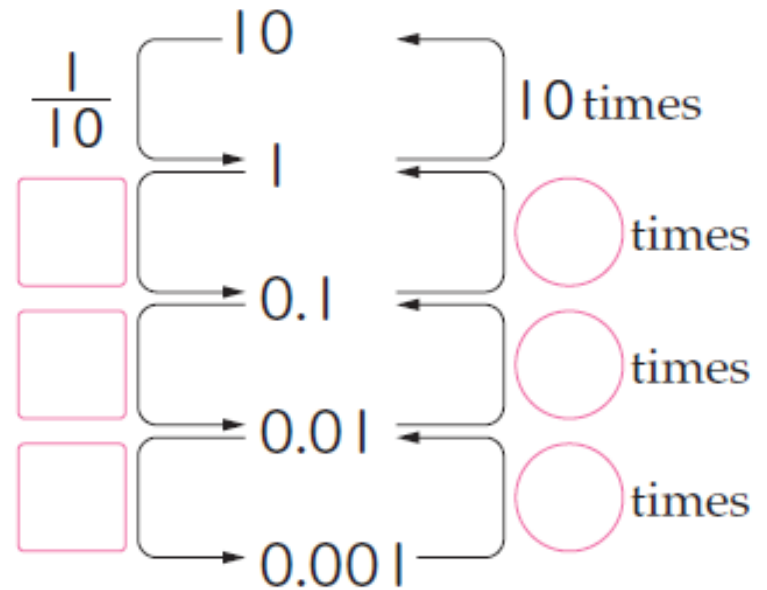
Positional and Non-positional System

5	五
55	<u>五十</u> 五
505	<u>五百</u> 五
5005	<u>五千</u> 五
50005	<u>五万</u> 五

Structure of Decimal Numeration

- ▶ Positional: where a numeral is written matters.
 - ▶ Each position (place) represents a specific value.
 - ▶ Adjacent positions (places) are always in 1 to 10 relationship – 10 of a smaller units make up 1 of the next larger unit.
- 

3 Write the number that goes in each of the on the right.
Also, write a number in each of the .



Multiplication/division by 10

3 If you multiply 25 10 times, then another 10 times, what number will you get?

1 How many times as much will it be if you multiply a number 10 times and then another 10 times?

25 $\xrightarrow{10 \text{ times}}$ 250 $\xrightarrow{10 \text{ times}}$ 2500

times

One thousands	Hundreds	Tens	Ones
		2	5
	2	5	0
2	5	0	0

$25 \times 10 \dots$
 $250 \times 10 \dots$

10 times
 10 times
 100 times

4 What number is 200 divided by 10?



200 $\xrightarrow{\text{Divided by } 10}$ 20

$200 \div 10 =$

Hundreds	Tens	Ones
2	0	0
	2	0

Divided by 10

Multiplication/division by 10

2

Investigate what happens to 3.75 when it is made 10 times and 100 times as much.

	Thousands place	Hundredths place	Tenths place	Ones place	Tens place	Hundreds place
		5	7	3		
10 times						
100 times						

3

Investigate what happens to 25.7 when it is made $\frac{1}{10}$ and $\frac{1}{100}$ as much.

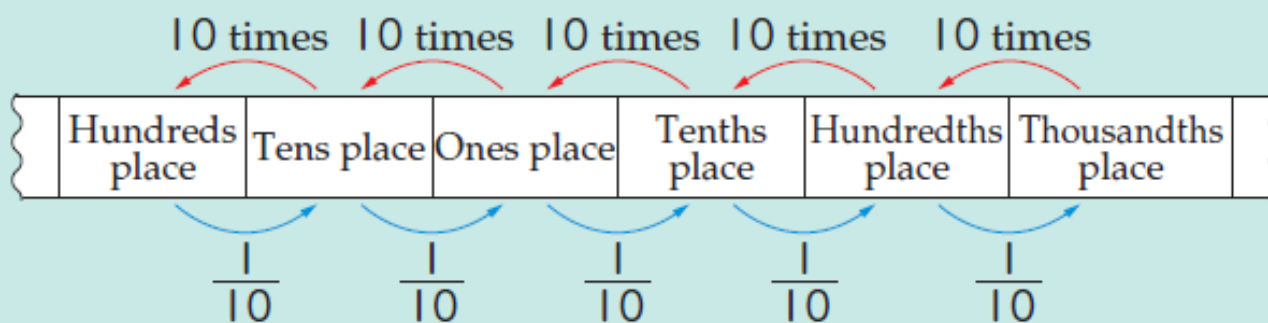
	Thousands place	Hundredths place	Tenths place	Ones place	Tens place	Hundreds place
			7	5	2	
$\frac{1}{10}$						
$\frac{1}{100}$						

Contrasting fractions and decimal numbers



The Differences Between the Structure of Whole / Decimal Numbers and Fractions

The place values for whole numbers and decimal numbers are always 10 times or $\frac{1}{10}$ of the adjacent place values.



"The place value is $\frac{1}{10}$ of the adjacent place" means that the new place value is one of 10 equally divided parts of the original place value, doesn't it?



Takumi

On the other hand, fractions are created to express amounts that are equally divided. Therefore, unlike whole numbers and decimal numbers the number of equal parts in a fraction does not have to be just 10.

Fractions were already being used in Egypt and Babylon (the southern part of today's Iraq) more than 4000 years ago, much earlier than decimal

Fractions were already being used in Egypt and Babylon (the southern part of today's Iraq) more than 4000 years ago, much earlier than decimal numbers. In Egypt, fractions with 1 as the numerator, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, ... were used to express one of 2, 3, 4, 5, ... equal parts.

In Babylon, they used the denominator of 60 and different numbers for the numerator. 60 was useful because it can be divided evenly by many whole numbers, and divided into equal parts in many different ways.



Shinji

One hour is 60 minutes. One whole turn is 360° I wonder if there is any relationship.

So we can see that the idea of dividing evenly is used differently in whole numbers, decimal numbers, and in fractions.

Relative Size of Numbers

- ▶ With the decimal numeration system, a number is represented as accumulation of units (powers of 10).
- ▶ 23.45 is made of
 - 2 units of 10
 - 3 units of 1
 - 4 units of 0.1
 - 5 units of 0.01

Relative sizes of 0.67

- ▶ 6 units of 0.1, and
- ▶ 7 units of 0.01

Or

- 67 units of 0.01

Or

- 670 units of 0.001

Or

- 6700 units of 0.0001

Or ...



Relative sizes of

▶ $\frac{2}{3}$ is made of 2 units of $\frac{1}{3}$

or

• $\frac{2}{3}$ is made of 4 units of $\frac{1}{6}$


or

• $\frac{2}{3}$ is made of 6 units of $\frac{1}{9}$

or ...

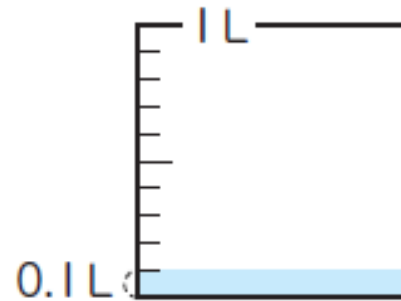


Structure of Decimal Numeration

- ▶ Positional: where a numeral is written matters.
 - ▶ Each position (place) represents a specific value.
 - ▶ Adjacent positions (places) are always in 1 to 10 relationship – 10 of a smaller units make up 1 of the next larger unit.
 - ▶ Each position must have one and only one numeral.
 - 0 as a place holder
 - Exceptions: leading 0's for whole numbers and trailing 0's for decimal numbers
- 

One and only one numeral in each place

- 3 What is the amount of water if there are nine 0.1 L?
Also, what is the amount of water if there are ten 0.1 L?



Tokyo Shoseki (2010) Gr.3 p. B15

2

What is the length of six $\frac{1}{5}$ m, seven $\frac{1}{5}$ m, and so on?

? Let's think about how to express fractions greater than 1.

The length of six $\frac{1}{5}$ m is expressed as $\frac{6}{5}$ m.

Gr.3 p. B49

Regularity in Repeated Reasoning

- ▶ Addition and subtraction of decimal numbers
- ▶ Thinking in terms of units other than 1

3 Addition and Subtraction of Decimal Numbers

- 1 A big bottle contains 0.5L of juice and a small bottle contains 0.3L of juice.
How much juice is there altogether?



- ? Let's think about how to calculate.

- 1 How many 0.1L are in 0.5L and 0.3L each?



If we think 0.1L as a unit,
 + so...

- 2 Explain how the calculation on the right was done.

$$0.8 + 0.2 = 1$$

- 2 Think about how to calculate $0.4 + 0.7$.

- 3 If you think of 0.1 as a unit, what kind of calculation does $0.4 + 0.7$ become?



0.4 is made of 0.1's, and
0.7 is made of 0.1's, so ...

- 3 There is 0.8L of juice.
She drank 0.3L of it.
How many L of juice are left?



- ? Let's think about how to calculate.

- 1 How many 0.1L are in 0.8L and 0.3L each?

If we think 0.1L as a unit,
 - so...



- 2 Explain how the calculation on the right was done.

$$1 - 0.4 = 0.6$$

- 4 Think about how to calculate $1.4 - 0.6$.

- 3 If you think of 0.1 as a unit, what kind of calculation does $1.4 - 0.6$ become?



1.4 is made of 0.1's, and
0.6 is made of 0.1's, so...

1

There is $\frac{3}{10}$ L of juice in a carton and $\frac{2}{10}$ L in a bottle. How much juice is the altogether in L?



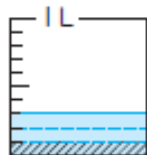
Write a math sentence.



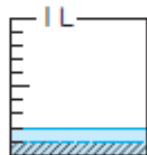
Let's investigate to find out if we can do addition with fractions.

Let's think about how to calculate

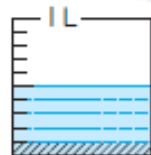
$$\frac{3}{10} + \frac{2}{10}$$



of $\frac{1}{10}$ L



of $\frac{1}{10}$ L

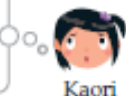


of $\frac{1}{10}$ L

$$\frac{3}{10} + \frac{2}{10} =$$



If they are decimal numbers, $0.3 + 0.2 = 0.5$, but...



Kaori



Think about it by looking at the fractions as how many $\frac{1}{10}$ L.

2

There is $\frac{4}{5}$ L of juice.

If a girl drinks $\frac{1}{5}$ L of juice, how much juice will be left in L?

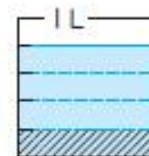


Write a math sentence.



Let's investigate to find out if we can do subtraction with fractions.

Let's think about how to calculate $\frac{4}{5} - \frac{1}{5}$.

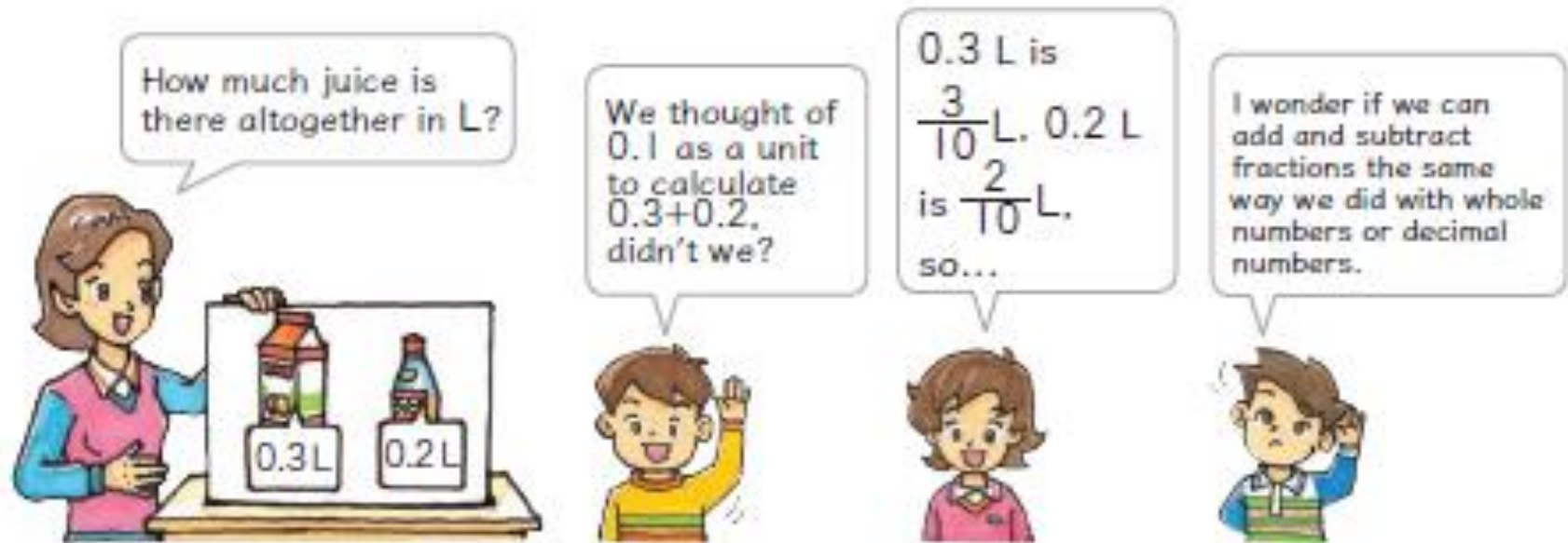


$$\frac{4}{5} - \frac{1}{5} =$$



If you think about it with $\frac{1}{5}$ L as a unit...

Emphasizing repeated reasoning



1

How many pieces of colored paper are there altogether?



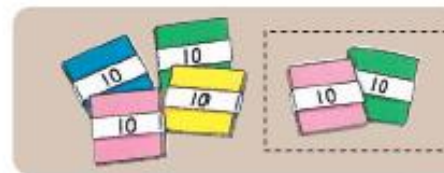
$$50 + 20 = \square$$

If we think about it in bundles of 10, there are 5 bundles and 2 bundles, so...



3

There are 60 pieces of colored paper.
If we use 20 pieces, how many will be left?



$$60 - 20 = \square$$

If we think about it in bundles of 10...



Tokyo Shoseki (2010) Gr.1 p. 128

3

Think about how to calculate $300 + 200$.

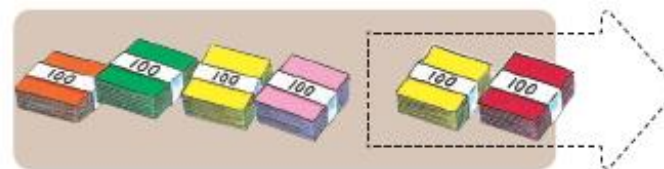


When we calculate tens, we think with bundles of ten so...



4

Think about how to calculate $600 - 200$.




★ Explain the calculation by thinking the same way as 3.

$$600 - 200 = \square$$

Tokyo Shoseki (2010) Gr.2 pp. A60 & A61

Multiplication of Decimal Numbers

- ▶ Multiplying decimal numbers by whole numbers in Grade 4
 - ▶ Multiplying by decimal numbers in Grade 5
 - ▶ CCSS 4.NF.4: multiplying fractions by whole numbers
 - ▶ CCSS 5.NF.4: multiplying by fractions
- 

Multiplying decimal numbers by whole numbers

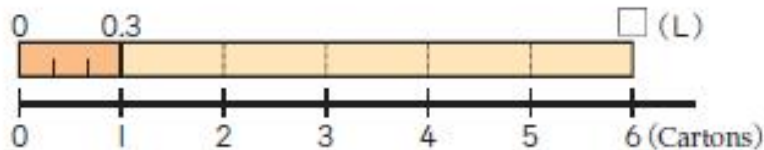
- ▶ Continue to make use of decimal units

1

We bought 6, 0.3L cartons of juice. How much juice is there altogether?



Tokyo Shoseki (2010) Gr.4 pp. B73 & B74



What math sentence should we write?



Let's think about how to calculate.

Can you explain the reason for the math sentence?

Explain the following two students' ideas.



Kaori

$0.3\text{L} = 3\text{dL}$
If we think in terms of dL as the unit,
 $3 \times 6 = 18$

$18\text{dL} = \boxed{} \text{L}$

Answer $\boxed{} \text{L}$



Hitoki



Since 0.3L is the amount made of 3 0.1L,
we can think in terms of 0.1 as the unit.

$3 \times 6 = 18$

18 0.1L together will be $\boxed{} \text{L}$.

Answer $\boxed{} \text{L}$



They are both changing the decimal number into a whole number, aren't they?

Multiplying decimal numbers by whole numbers

► Make use of property of multiplication

- ★ Think about how to calculate
(decimal number) \times (whole number)
based on the way we calculate
(whole number) \times (whole number).

$$\begin{array}{ccc} 0.3 \times 6 = 1.8 & \xrightarrow{\text{Divide by } 10} & 18 \\ \downarrow \text{10 times} & & \downarrow \text{10 times} \\ 3 \times 6 = 18 & & \end{array}$$

The product of 0.3×6 can be calculated by first making 0.3 10 times as much, then by calculating 3×6 , and then by dividing the product by 10.

Tokyo Shoseki (2010) Gr.4 p. B74

The answer for 5×30 is the same as 10 times as much as 5×3 . Therefore, the answer is the same as placing a 0 to the right of 15.

$$\begin{array}{ccc} 5 \times 3 = 15 & & \\ \downarrow 10 \text{ times} & \downarrow 10 \text{ times} & \\ 5 \times 30 = 150 & & \end{array}$$



When the number in the multiplier becomes 10 times as much, the answer also becomes 10 times as much.

Tokyo Shoseki (2010) Gr.3 p. B64

Multiplying by decimal numbers

- ▶ Making sense of multiplication by decimal numbers first

Tokyo Shoseki (2010) Gr.5 pp. A31 & A32

1 1 meter of ribbon costs 80 yen. I bought 2.3m of the ribbon, how much was the cost?



Let's think about what math sentence we should write.



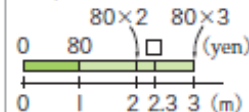
If it were 3m, we could think of it as three times the cost of 1m of ribbon, but...



Price for 1m \times Length bought = Cost



If we buy 2m or 3m, the cost will be 2 and 3 times the price for 1m, so...

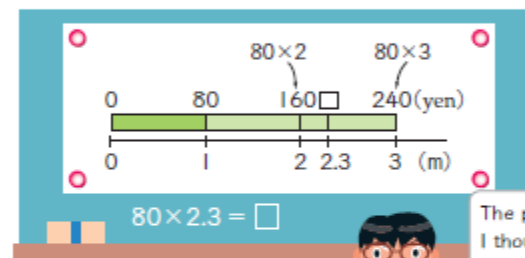


1 Explain why the math sentence is written in this way.

I thought we could think of the lengths as if they were whole numbers.



Price for 1m		\times	Length bought	=	Cost
2m	80	\times	2	=	160
3m	80	\times	3	=	240
2.3m	80	\times	2.3	=	<input type="text"/>



The price for 1m is 80 yen. I thought the cost for 2.3m should be 2.3 times 80 yen, and that's why I thought we could use multiplication.

Even when the length of ribbon is a decimal number, we can use a multiplication sentence to find the total cost, just like we did when the lengths were whole numbers.

Summary

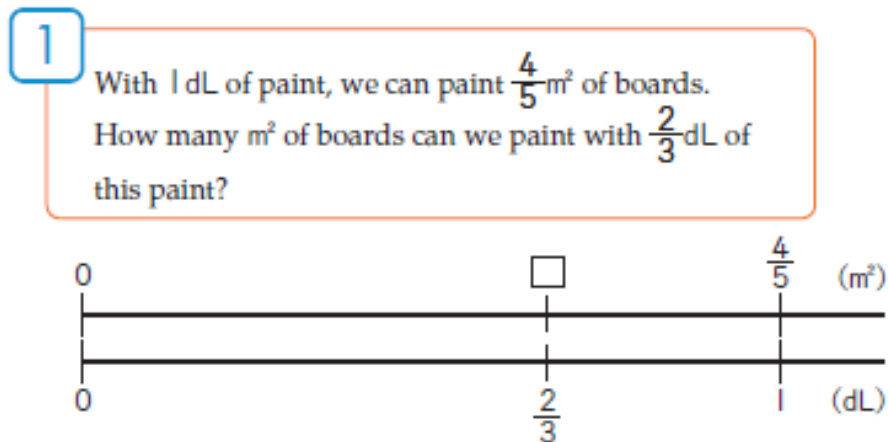
80×2.3



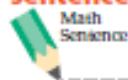
About how much will it be? It will be greater than 80×2 , but 80×3 ...

Multiplying by fractions

- ▶ Making sense of multiplication by fractions first



? Let's think about what math sentence we should write.



With 2dL, we can think of it as 2 of the amount that can be painted with 1dL, but with $\frac{2}{3}\text{dL}$...

Area we can paint with 1 dL	\times	Amount of paint (dL)	$=$	Area we can paint
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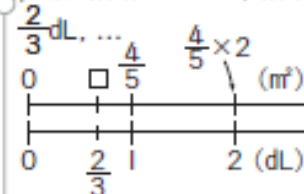


Shinji



Kaori


The amount of boards we can paint with 2dL will be 2 times the area we can paint with 1dL. So, with $\frac{2}{3}\text{dL}$, ...



★ Explain the reason for your math sentence.

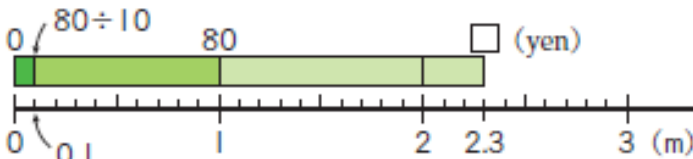
Ways to multiply by decimal numbers

- ▶ Thinking in terms of decimal units
- ▶ Using property of multiplication

 Takumi

2.3m is made up of 23 0.1m pieces.
So, we can find the price for 0.1m,
and then find what 23 times that price is.

$$\begin{array}{ccc} & \div 10 & \\ \square \text{ yen} & & 80 \text{ yen} \\ & \div 10 & \\ 0.1 \text{ m} & & 1 \text{ m} \end{array}$$




- Price of 0.1m... $80 \div 10$
- Cost of 2.3m... $(80 \div 10) \times 23$

$$80 \times 2.3 = 80 \div 10 \times 23$$


$$= \square$$

Answer yen

 Kaori

If the length of the ribbon becomes
10 times as long, the cost will also
be 10 times as much.

$$\begin{array}{ccc} 80 \times 2.3 = \square & & \\ \downarrow \times 10 \downarrow \times 10 & & \div 10 \\ 80 \times 23 = 1840 & & \end{array}$$



- Cost of 23m... 80×23
- Cost of 2.3m... $(80 \times 23) \div 10$

$$80 \times 2.3 = 80 \times 23 \div 10$$

$$= \square$$

Answer yen

Ways to multiply by fractions

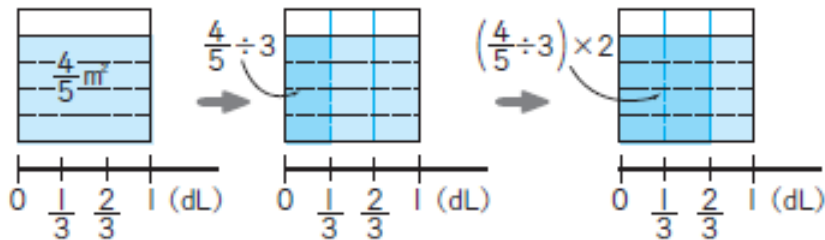
- ▶ Thinking in terms of fraction units
- ▶ Using property of multiplication



Yumi

First, find the area of boards you can paint with $\frac{1}{3}$ dL, and then double that amount.

(Area we can paint with 1 dL) (Area we can paint with $\frac{1}{3}$ dL) (Area we can paint with $\frac{2}{3}$ dL)



$$\begin{aligned}\frac{4}{5} \times \frac{2}{3} &= \left(\frac{4}{5} \div 3\right) \times 2 \\ &= \frac{4}{5 \times 3} \times 2 \\ &= \square \times \square \\ &= \square \times \square \\ &= \square\end{aligned}$$



Hiroki

If we change $\frac{2}{3}$ into a whole number, we can calculate it. We make the multiplier 3 times as much, and then divide the product by 3.

$$\frac{4}{5} \times \frac{2}{3} = \frac{4}{5} \times \left(\frac{2}{3} \times \frac{1}{1}\right) \div 3$$

$$= \frac{4}{5} \times 2 \div 3$$

$$= \square \times \square$$

$$= \square \times \square$$

$$= \square$$

$$\begin{aligned}\frac{4}{5} \times \frac{2}{3} &= \square \\ \frac{4}{5} \times \left(\frac{2}{3} \times \frac{1}{1}\right) &= \frac{4}{5} \times 2 \div 3\end{aligned}$$

$$80 \times 2.3 = 184$$

$$80 \times 23 = 1840$$

It's the same thinking we used with decimal numbers, isn't it?

Division of decimal numbers

- ▶ Dividing decimal numbers by whole numbers in Grade 4
- ▶ Dividing by decimal numbers in Grade 5

Dividing decimal numbers by whole numbers

- ▶ Making use of context
- ▶ Making use of decimal units

1

We bought 3.6L of water. If we share this water equally among 3 people, how much water will each person get?



1

What math sentence should we write?



Can you explain the idea behind why you wrote this math sentence?

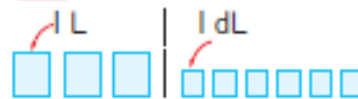
2

Explain the following two students' ideas.



Takumi

We split 3.6L into L and dL.



$$\square \div 3 = \square \text{ (L)}$$

$$\square \div 3 = \square \text{ (dL)}$$

Together, L.

$$3.6 \div 3 = \square$$



Kaori

3.6L is made of 0.1L.

$$\square \div 3 = \square$$

Each person get 0.1L, or L.

Answer L

?

Let's think about how to calculate.

Dividing decimal numbers by whole numbers

- ▶ What if the dividend is not evenly divisible?
 - Remainder
 - Dividing on

7

Calculate $46.7 \div 3$ using the division algorithm. Calculate the quotient to the ones place, and find the remainder.



Let's think about the size of the remainder when we divide decimal numbers.

8

If we share 6L of juice equally among 4 people, how much juice will each person get?



$6 \div 4 = 1$
remainder 2, but ...

6L is made of 60
0.1L, so, ...

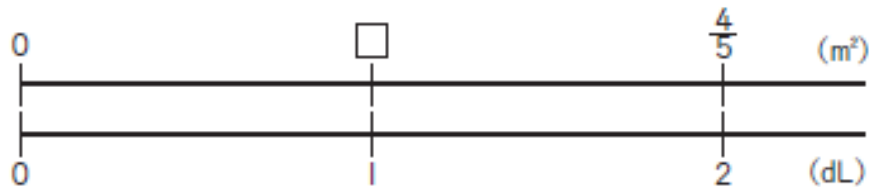


Let's think about how to continue dividing with the division algorithm.

Dividing fractions by whole numbers

3

You can paint $\frac{4}{5} \text{ m}^2$ of boards with 2 dL of paint.
How many m^2 can you paint with 1 dL of this paint?



4

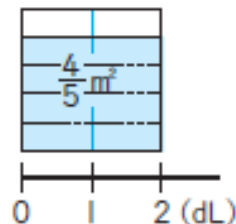
Think about how to calculate $\frac{4}{5} \div 3$.

★ What math sentence do we need to write?



Can you explain your answer?

❓ Let's think about how to calculate.



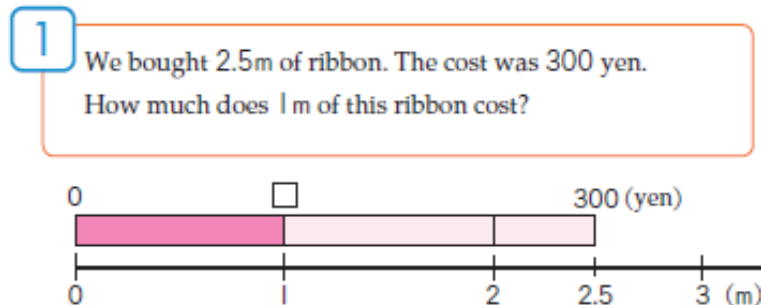
$4 \div 3$ cannot be divided completely, so ...

I wonder if we can change $\frac{4}{5}$ into another fraction that has a numerator that can be divided by 3 ...



Dividing by decimal numbers

- ▶ Making sense of multiplication by decimal numbers first

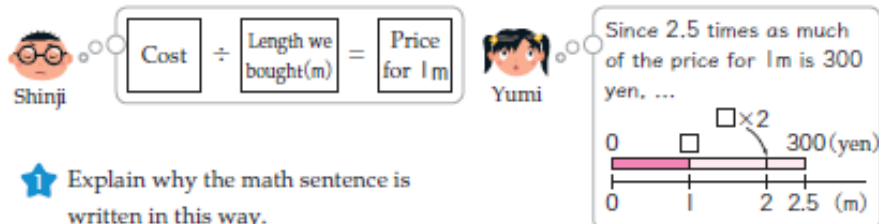


? Let's think about what math sentence we should write.

Math Sentence

If it were 3m, then, we could think of it as 3 pieces of 1m and divide 300 yen into 3 equal pieces. But, ...

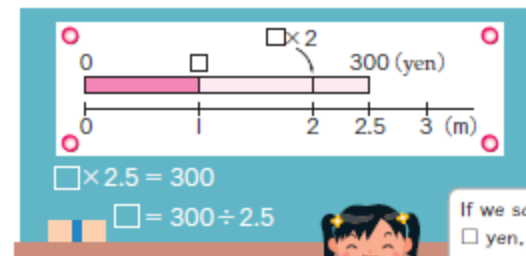
Kaori



If the length we bought was a whole number, it will be division, ...



Cost	÷	Length we bought(m)	=	Price for 1m
2m ... 300	÷	2	=	150
3m ... 300	÷	3	=	100
2.5m ... 300	÷	2.5	=	□



If we say the price for 1m is □ yen, the math sentence will be □ × 2.5 = 300. Since we are trying to find the value of □, I thought it should be 300 ÷ 2.5.

Even when the length of ribbon is a decimal number, we can use division to find the price for 1m just like we did with whole numbers.

Summary

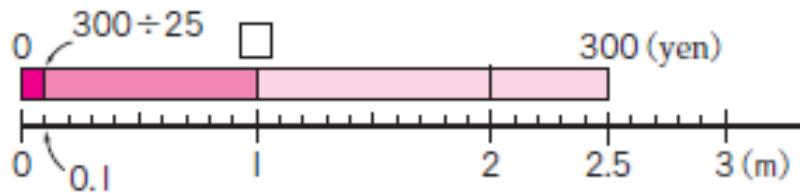
Ways to divide by decimal numbers

- ▶ Thinking in terms of decimal units
- ▶ Using property of division



Miho

2.5m is made up of 25 pieces of 0.1 m.



- Price for 0.1 m..... $300 \div 25$
- Price for 1 m $(300 \div 25) \times 10$

$$300 \div 2.5 = 300 \div 25 \times 10$$

$$= \boxed{}$$

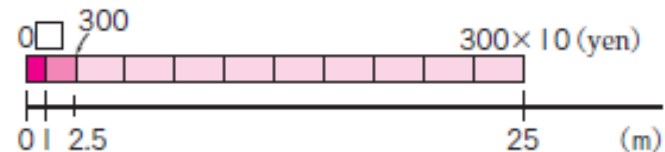
Answer $\boxed{}$ yen



Hiroki

If the length of the ribbon becomes 10 times as long, the cost will be 10 times as much, but the price for 1 m does not change.

$$\begin{array}{l} 300 \div 2.5 = \boxed{} \\ \downarrow \times 10 \quad \downarrow \times 10 \\ 3000 \div 25 = 120 \end{array} \quad \text{equal}$$



- Cost of 25m..... 300×10
- Price for 1 m..... $(300 \times 10) \div 25$


$$300 \div 2.5 = 300 \times 10 \div 25$$

$$= \boxed{}$$

Answer $\boxed{}$ yen

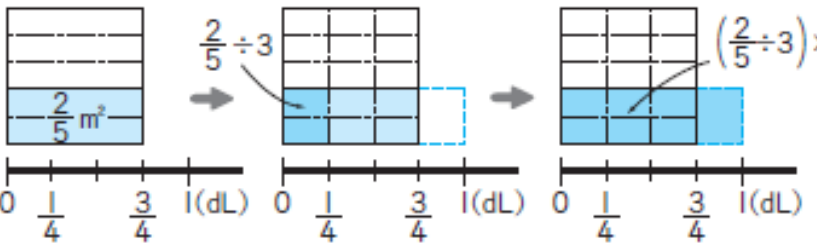
Ways to divide by fractions

- ▶ Thinking in terms of fraction units
- ▶ Using property of division


 Kaori

First, find how much area can be painted with $\frac{1}{4}$ dL, and then find 4 times as much as that number.

(Area painted by $\frac{3}{4}$ dL) (Area painted by $\frac{1}{4}$ dL) (Area painted by 1 dL)



$\frac{2}{5} \div 3 = \left(\frac{2}{5} \div 3\right) \times 4$
 $\frac{2}{5} \div \frac{3}{4} = \left(\frac{2}{5} \div 3\right) \times 4$
 $= \frac{2}{5 \times 3} \times 4$
 $= \frac{\square \times \square}{\square \times \square}$
 $= \square$

 Shinji


We can calculate if we can change $\frac{3}{4}$ into a whole number...

$$\frac{2}{5} \div \frac{3}{4} = \square$$

$$\left(\frac{2}{5} \times 4\right) \div \left(\frac{3}{4} \times 4\right) = \frac{2}{5} \times 4 \div 3$$

equal

$\frac{2}{5} \div \frac{3}{4} = \left(\frac{2}{5} \times 4\right) \div \left(\frac{3}{4} \times 4\right)$
 $= \left(\frac{2}{5} \times 4\right) \div 3$
 $= \frac{2 \times 4}{5} \div 3$
 $= \frac{\square \times \square}{\square \times \square}$
 $= \square$




$200 \div 2.5 = 80$
 $\left[\times 10 \right] \left[\times 10 \right]$
 $2000 \div 25 = 80$


equal

It's the same idea we used when we divided by a decimal number, isn't it?

Final Thoughts

- ▶ In order for students to look for and make use of structures in learning of decimal numbers, structures must become a focus in their learning of whole numbers (and fractions).
 - ▶ It is helpful to have a curriculum flow that makes use of structures as a theme.
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Final Thoughts

- ▶ In order for students to look for and express regularity in repeated reasoning with decimal numbers, reasoning must become a focus in mathematics lessons.
 - ▶ Tasks for lessons must be carefully chosen so that desired reasoning is more likely to arise from students.
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Thank you!