Teaching Proportional Relationships: A Japanese Perspective

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Conclusions

- It might be worth investigating the possibility of introducing proportional relationships early – even before students learn ratios/rates
- Learning of proportional relationship should enhance students' capacity to examine situations mathematically
- We might want to re-consider the purposes of teaching/learning proportional relationships.

Ratio/rate/proportional relationships – important but difficult to teach/learn

- “proportional reasoning is the capstone of children’s elementary school arithmetic and the cornerstone of all that is to follow” (Lesh, Post, and Behr, 1988, pp. 93-94).
- Proportionality is described to be “of such great importance that it merits whatever time and effort must be extended to assure its careful development” (NCTM 1989, p. 82)
- “(o)f all the topics in the school curriculum, fractions, ratios, and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites” (Lamon, 2007, p. 629).

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Georgia's K-12 Mathematics Standards
Mathematics Big Ideas and Learning Progressions, K-12

<table>
<tr><th>K</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>HS Algebra: Concepts & Connections</th><th>HS Geometry: Concepts & Connections</th><th>HS Advanced Algebra: Concepts & Connections</th></tr>
<tr><td colspan="12">Mathematical Modeling (MM)</td></tr>
<tr><td colspan="12">Mathematical Practices (MP)</td></tr>
<tr><td colspan="12">Data & Statistical Reasoning (DSR)</td></tr>
<tr><td colspan="12">Numerical Reasoning (NR)</td></tr>
<tr><td colspan="12">Patterning & Algebraic Reasoning (PAR)</td></tr>
<tr><td colspan="12">Geometric and Spatial Reasoning (GSR)</td></tr>
<tr><td colspan="6">Measurement & Data Reasoning (MDR)</td><td colspan="6"></td></tr>
<tr><td colspan="8"></td><td colspan="4">Functional & Graphical Reasoning (FGR)</td></tr>
<tr><td colspan="7"></td><td>Probability Reasoning (PR)</td><td colspan="2"></td><td colspan="2">Probabilistic Reasoning (PR)</td></tr>
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K-12 MATHEMATICS LEARNING PROGRESSION - GEORGIA

Key Concepts	ELEMENTARY SCHOOL (K-5)						MIDDLE SCHOOL (6-8)			HIGH SCHOOL (9-12)			
	K	1	2	3	4	5	6	7	8	Algebra: Concepts & Connections	Geometry: Concepts & Connections	Advanced Algebra: Concepts & Connections	Courses beyond Advanced Algebra
Ratios & Rates	*In the early years, students are building foundational knowledge by acquiring a conceptual understanding of fractions and decimals. This knowledge will be applied to the concept of ratios and rates in middle school.*						**Numerical Reasoning with ratios and rates:** • Concept of ratio and rate • Equivalent ratios, percents, unit rates • Convert within measurement systems	• Compute unit rates associated with ratios of fractions • Determine unit rates	• Interpret unit rate as the slope of a graph	• Convert units and rates given a conversion factor	• Side ratios of similar triangles • Trigonometric ratios	• Average rate of change of quadratic, exponential, logarithmic, and radical functions • Trigonometric ratios	• Apply the concept of ratio and rate reasoning to solve contextual, real-life, mathematical problems
Proportional Relationships	*In the early years, students are building foundational knowledge by acquiring a conceptual understanding of fractions and decimals. This knowledge will be applied to the concept of proportional relationships later.*						*In Grade 6, students should develop a foundation for understanding proportions through the development of ratio and rate reasoning, as well as part-whole computational strategies related to fractions, decimals, and percents.*	• Use proportional relationships • Solve multi-step ratio and percent problems • Scale drawings of geometric figures • Use similar triangles to explain slope	*In Grade 8, students should extend their understanding of proportions to derive the equation $y = mx + b$.*	• Apply the concept of proportionality to functions and their graphs	• Apply the concept of proportionality to functions and their graphs	• Apply the concept of proportionality to functions and their graphs	• Apply the concept of proportionality to functions and their graphs

K - 5

In the early years, students are building foundational knowledge by acquiring a conceptual understanding of fractions and decimals. This knowledge will be applied to the concept of proportional relationships later.

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GA Learning Progression

Grade 6

In Grade 6, students should develop a foundation for understanding proportions through the development of ratio and rate reasoning, as well as part-whole computational strategies related to fractions, decimals, and percents.

Grade 7

- *Constant of proportionality*
- *Use proportional relationships*
- *Solve multi-step ratio and percent problems*
- *Scale drawings of geometric figures*
- *Use similar triangles to explain slope*

Grade 8

In Grade 8, students should extend their understanding of proportions to derive the equation $y = mx + b$.

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Ratio/Rate/Proportional Relationships in the Japanese National Course of Study

Grade	Topics
4	Relationships of two co-varying quantities
5	Average/Per-unit quantity Percentage/*Wariai* (ratio of two quantities as a measure) Simple proportional relationships
6	Ratio Direct and inversely proportional relationships
7	Direct and inversely proportional relationships

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Grade 4: Relationships of co-varying quantities

In this unit, the phrase “proportional relationship” does not appear. However, the Japanese COS positions the study of proportional relationships in the domain of “changes and relationships” a domain for Grades 4 through 6.

In the previous COS, the study of proportional relationships was positioned within the study of functional relationships, i.e., proportional relationships as specific functional relationships.

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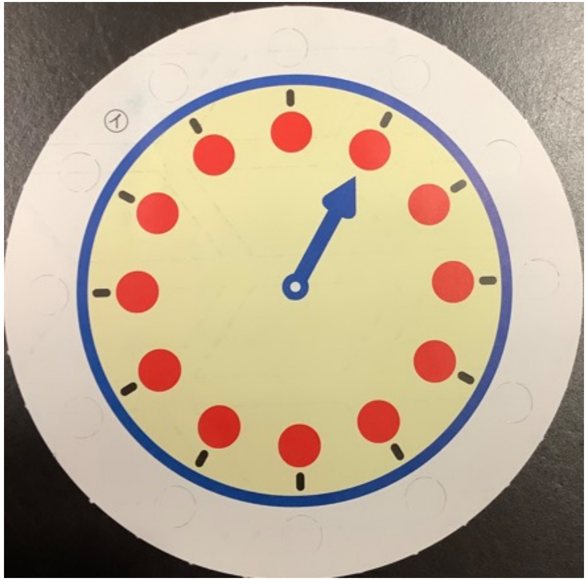
Opening Problem: Curious Clocks

Front

Back



Front	1	2	3	4	5	6	7	8
Back								

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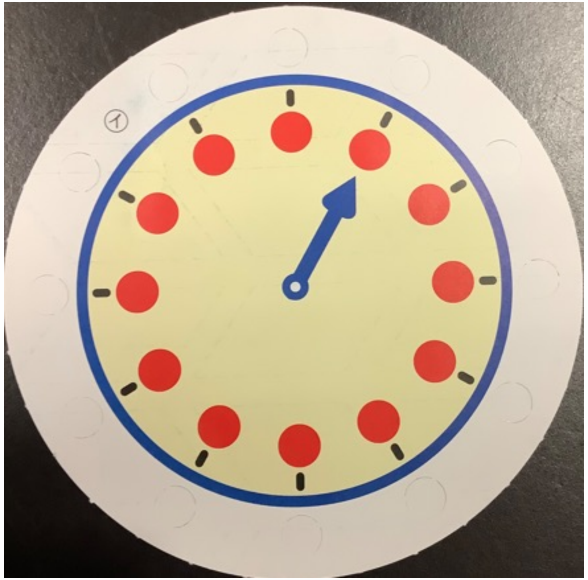
Opening Problem: Curious Clocks

Front

Back



Front	1	2	3	4	5	6	7	8
Back	12	11	10	9	8	7	6	5

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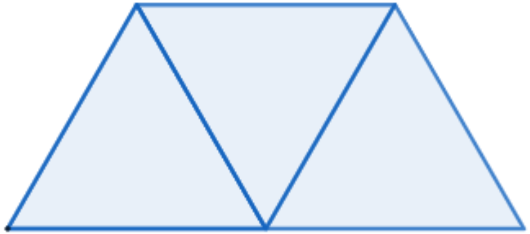
Problem 2: # of triangles and the perimeter



Arrange equilateral triangles with 1-cm sides in a straight row.
Find the perimeter when 20 triangles are arranged.

# of triangles	1	2	3	4	5	6	7
Perimeter							

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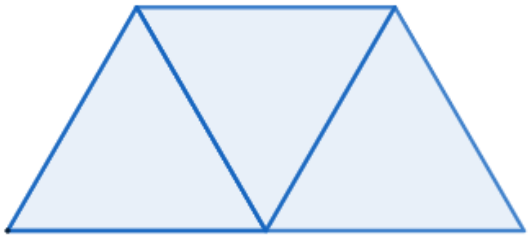
Problem 2: # of triangles and the perimeter



Arrange equilateral triangles with 1-cm sides in a straight row.
Find the perimeter when 20 triangles are arranged.

# of triangles	1	2	3	4	5	6	7
Perimeter	3	4	5	6	7	8	9

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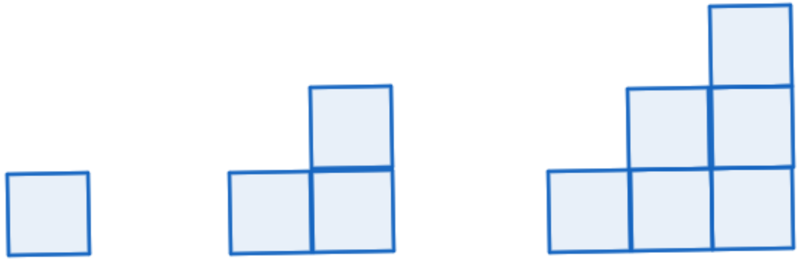
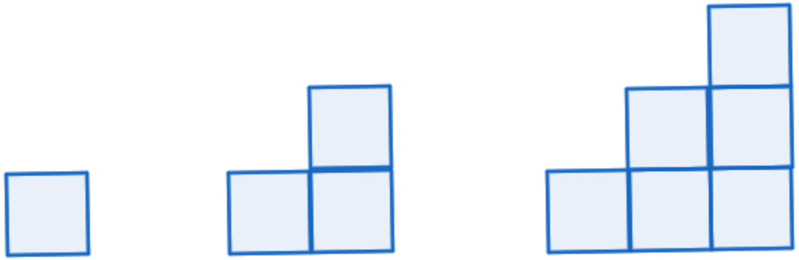
Problem 3: # of ‘steps’ and the perimeter



Make a staircase using squares with 1-cm sides. What is the perimeter when there are 20 steps.

# of steps	1	2	3	4	5	6	7
Perimeter							

Problem 3: # of ‘steps’ and the perimeter



Make a staircase using squares with 1-cm sides. What is the perimeter when there are 20 steps.

# of steps	1	2	3	4	5	6	7
Perimeter	4	8	12	16	20	24	28

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Key Ideas with Problems 1 - 3

- Summarize the corresponding values of the two quantities in a table.
- Explore the relationship that exists between two quantities.

Reading a table horizontally & vertically

+ 1 + 1

# of steps	1	2	3	4	5	6	7
Perimeter	4	8	12	16	20	24	28

?

# of steps	1	2	3	4	5	6	7
Perimeter	4	8	12	16	20	24	28

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Key Ideas with Problems 1 - 3

- Summarize the corresponding values of the two quantities in a table.
- Explore the relationship that exists between two quantities.
- Express the relationship using an equation, using symbols (□ and ○).

  □ + ○ = 13 (Problem 1)

  □ + 2 =○, or ○ - □ = 2 (Problem 2)

  □ × 4 =○ (Problem 3)

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An extra consideration with Problem # 3

Reading a table horizontally differently:

4 times

3 times

2 times

# of steps	1	2	3	4	5	6	7
Perimeter	4	8	12	16	20	24	28

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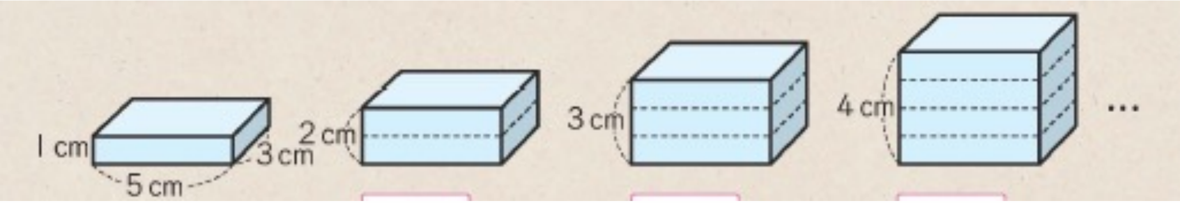
Grade 5: Let's explore how quantities change



When the height of a rectangular prism changes 1cm, 2cm, 3cm, …how does the volume change?

Height (□ cm)	1	2	3	4	5	6	7	8
Volume (○ cm^3)	15							

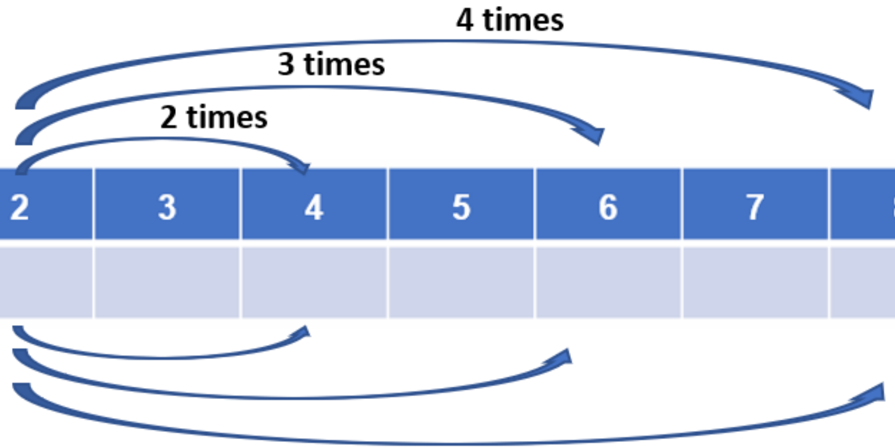
Reading the table

4 times
3 times
2 times

Height (□ cm)	1	2	3	4	5	6	7	8
Volume (○ cm^3)	15							

Starting with the height of 1 cm, explore how the volume changes as the height becomes 2 times, 3 times, 4 times, … as much.

Reading the table



Height (□ cm)	1	2	3	4	5	6	7	8
Volume (○ cm^3)	15							

Starting with the height of 2 cm, explore how the volume changes as the height becomes 2 times, 3 times, 4 times, … as much.

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1st definition of proportional relationship

There are two co-varying quantities □ and ○. If as □ becomes 2, 3, … times as much ○ also becomes 2, 3, … times as much, then we say that ○ is proportional to □.

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Practice Problems

Emphasis on distinguishing proportional situations from non-proportional situations using the definition.

Does each of the following show a pair of quantities that are in a proportional relationship?

1. Buying □ pieces of 25-yen construction paper and the total price, ○ yen.
2. Buying □ pieces of 25-yen construction paper along with one 50-yen eraser and the total price, ○ yen.
3. A rectangle with the vertical side of 4 cm and the horizontal side of □ cm, and its area○ cm^2.

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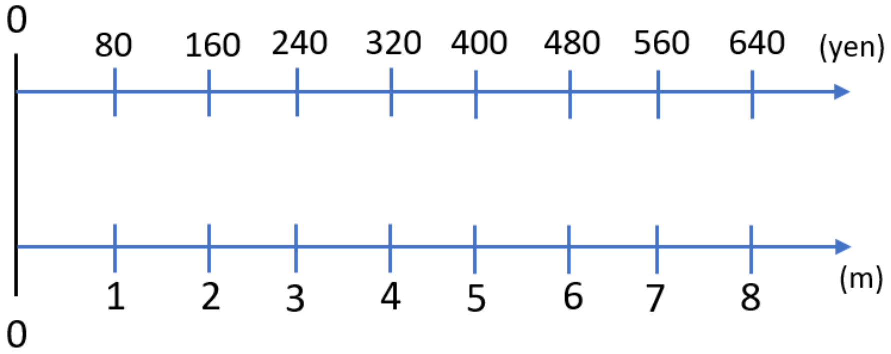
A new representation: Double Number Line

1 meter of a ribbon costs 80 yen. If the length of the ribbon changes 1 m, 2 m, 3 m, …, how does the total price change?

Length (□ m)	1	2	3	4	5	6	7	8
Price (○ yen)	80	160	240	320	400	480	560	640



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Grade 5 Applications of PR

Multiplication by decimal numbers

1 meter of a ribbon costs 80 yen. I bought 2.3 m of the ribbon.
How much was the cost?

What calculation is needed?

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Grade 5 Applications of PR

Multiplication by decimal numbers

1 meter of a ribbon costs 80 yen. I bought 2.3 m of the ribbon. How much was the cost?

One possible approach:

The cost of 23 m will be 10 times of the cost of 2.3 m.

The cost of 23 m will be 80 × 23.

The cost of 2.3 m will be (80×23)÷10.

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Grade 6: In-depth study of PR

Unit 10: Direct and inverse proportional relationships

- Unit 2: Using letters for unknowns
- Unit 3: Multiplication of fractions
- Unit 4: Division of fractions
- Unit 5: Ratios
- Unit 6: Scale drawings

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Grade 6: Opening Problem

Water is poured into an aquarium from a faucet. The relationship between the amount of time water is poured into (*x* minutes) and the depth of water (*y* cm) are given in a table.

Time (*x* min)	1	2	3	4	5	6	
Depth (*y* cm)	4	8	12	16	20	24	

Time and Depth are in a proportional relationship.

Students were to investigate the relationship, in particular how values of *x* and *y* change.

Grade 6: Opening Problem

Since students have now learned to use decimals and fractions with "____ times as many/much," students are asked explicitly to think about those relationships:

Time (*x* min)	1	2	3	4	5	6	
Depth (*y* cm)	4	8	12	16	20	24	

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Grade 6: Extending the definition

If *x* and *y* are in a proportional relationship, then if *x* becomes ☐ times as much, then *y* will also become ☐ times as much.

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Grade 6: Reading a table vertically

Time (x min)	1	2	3	4	5	6	
Depth (y cm)	4	8	12	16	20	24	

x	1	2	3	4	5	6	
y	4	8	12	16	20	24	
$y \div x$							

If y is proportional to x, then the quotient of $y \div x$ is constant. The relationship can be expressed as

$$y = (constant\ number) \times x$$

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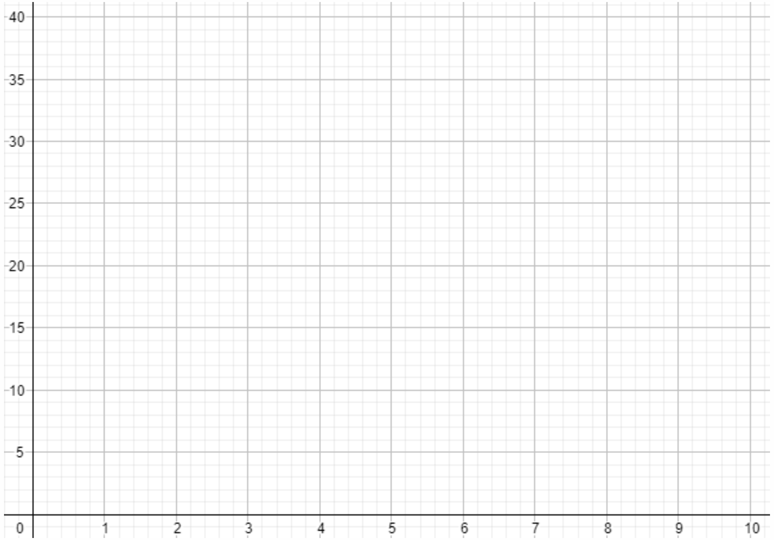
A new representation: A graph of PR

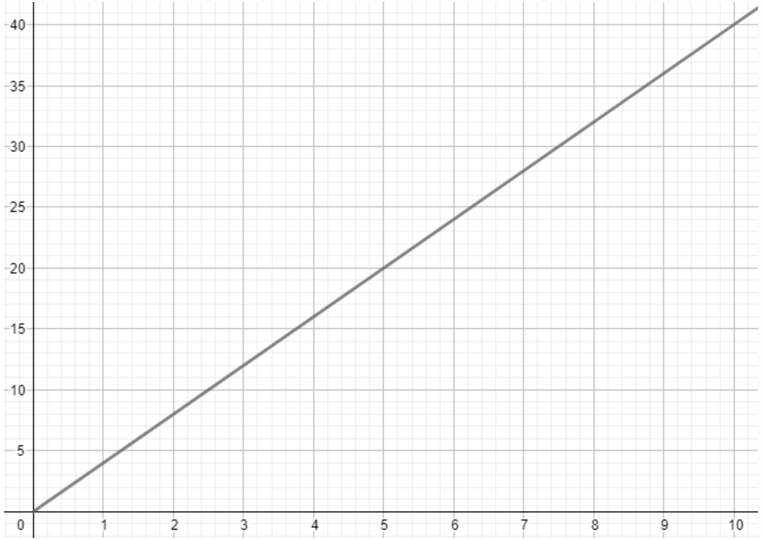
Time (*x* min)	1	2	3	4	5	6	
Depth (*y* cm)	4	8	12	16	20	24	



Plot the values on the graph by using the horizontal axis for *x* and the vertical axis for *y*.

A new representation: A graph of PR

Time (*x* min)	1	2	3	4	5	6	
Depth (*y* cm)	4	8	12	16	20	24	



The graph of a two quantities in a proportional relationship will be graphed as a straight line that goes through 0 (the origin).

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Grade 6: Inverse proportional relationship

Explore how values of *y* change as the values of *x* change in the following situations:

(A) If you walk at the speed of *x* km per hour, it takes *y* hours to walk a 6 km path.

(B) The base, *x* cm, and the height, y cm, of the parallelogram with the area of 12 cm^2.

(C) If we pour water into a depth 60 cm aquarium at the rate of *x* cm of water per minute, it will take *y* minutes to fill up the aquarium.

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Grade 6: Inverse proportional relationship

Definition: If as the values of *x* becomes 2, 3, 4, … times as much, the values of *y* becomes $\frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ … times as much, we say *y* is inversely proportional to *x*.

When *x* and *y* are in an inversely proportional relationship, the product of corresponding values of *x* and *y* are constant:

$$x \times y = constant$$

A graph of an inversely proportional relationship will be a curve.

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Grade 7: Direct and Inversely PR

Unit 4

- Section 1: Functions and direct/inversely proportional relationships
- Section 2: Properties of direct proportional relationships and ways to investigate them
- Section 3: Properties of inversely proportional relationships and ways to investigate them
- Section 4: Applications of direct/inversely proportional relationships

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Grade 7: Unit 4 Opening Problem

How long will it take to fill up a school pool?

Equivalent to the opening problem in Grade 6 unit on PR.

Define what a function is:

> When there are two variables, *x* and *y*, if the value of *y* is determined when the value of *x* is fixed, then we say *y* is a function of *x*.

Grade 7 Unit 4

Section 1.2 "Let's look back on direct and inversely proportional relationships we learned in elementary schools."

Revising the definition:

If *y* is a function of *x* and their relationship can be expressed in the following equation,

$$y = ax,$$

we say *y* is proportional to *x*.

a is called the constant of proportion.

Proportional relationships are functions.

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Grade 7 Unit 4

Section 2: Properties of PR and ways to investigate them

- Expanding the values of *x* and *a* to negative numbers
- Graphs of proportional relationships using all 4 quadrants

Graphs of proportional relationships are line through the origin.

The value of *a* shows how much *y* increases as *x* increases by 1, and it is also the quotient of $y \div x$. The value of *a* is also the value of *y* when *x* = 1.

If *a* > 0, the graph is increasing and if *a* < 0, the graph is decreasing.

"slope" is not used – "rate of change" and "slope" are introduced in Grade 8 unit on linear functions.

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When do students learn to “solve proportions”?

Unit 3: Let’s think about figuring out how to find unknown numbers

Section 1: Equations and how to solve them

1. Equations and their solutions
2. How to solve equations
3. Various equations

Section 2: Applications of equations

1. Applications of linear equations
2. Applications of proportion equations

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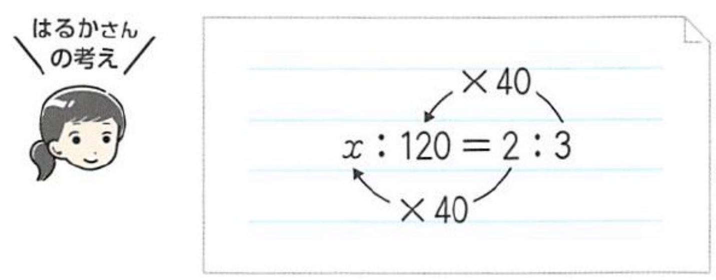
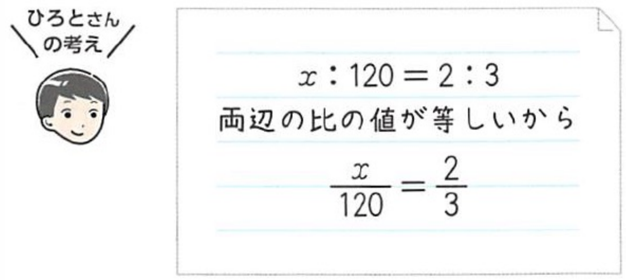
When do students learn to "solve proportions"?

Proportion equations

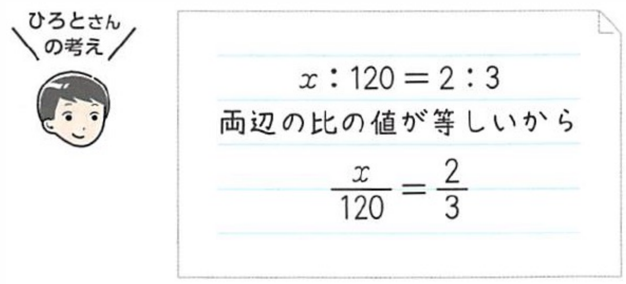
In the form of $a : b = c : d$.

How to solve them: Ex. $x : 120 = 2 : 3$





When do students learn to “solve proportions”?



This is an equation – based on the fact that the “values of ratios” on the right and left sides are equal.
Solve the equation using the methods students have learned.

Alternately: Derive an equation based on the property of proportion equations,

$$a : b = m : n \leftrightarrow an = bm$$

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Conclusions

- It might be worth investigating the possibility of introducing proportional relationships early – even before students learn ratios/rates using a friendly definition for lower grades.
- Learning of proportional relationship should enhance students' capacity to examine situations mathematically – viewing a table 'vertically' vs. 'horizontally; viewing changes additively vs. multiplicatively, double number lines, etc.
- We might want to re-consider the purposes of teaching/learning proportional relationships.
  - Foundation for understanding functions → correspondences and co-variations; expressing relationships in equations; etc.
  - PRs are particular instances of linear functions
  - Solving proportions → solving linear equations
  - Opportunities to examine multiplication/division from a new perspective.

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Thank you!

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