1. (8 points) Verify that the function $y = e^{3x} \cos(2x)$ is a solution of the differential equation

$$y'' - 6y' + 13y = 0$$

on the interval $(-\infty, \infty)$.

$$y' = -2e^{3x} \sin(2x) + 3e^{3x} \cos(2x)$$

$$y'' = -4e^{3x} \cos(2x) - 6e^{3x} \sin(2x) - 6e^{3x} \sin(2x) + 9e^{3x} \cos(2x)$$

$$y'' - 6y' + 13y = 5e^{3x} \cos(2x) + 13 \frac{3e^{3x} \cos(2x) - 12e^{3x} \sin(2x)}{x}$$

$$= 0 \quad \text{for all} \quad x \in (-\infty, \infty)$$

2. (10 points) Given a two-parameter family of solutions $x = c_1 \cos t + c_2 \sin t$ of the differential equation $x'' + x = 0$, find a solution of the IVP consisting of the differential equation and the initial conditions $x(\pi/6) = \frac{1}{2}$, $x'(\pi/6) = 0$.

$$c_2 = \frac{\sqrt{3}}{2}$$

$$c_1 = \frac{1}{2} + c_2 \frac{\sqrt{3}}{2}$$

$$c_1 + \frac{\sqrt{3}}{2} c_2 = 1$$

$$c_1 + \sqrt{3} c_2 = 1$$

$$c_2 = \frac{\sqrt{3}}{2}$$

$$c_1 = \frac{1}{4}$$

$$c_2 = \frac{1}{4}$$

The solution is

$$x = \frac{1}{4} \cos t + \frac{\sqrt{3}}{4} \sin t$$

$$\frac{1}{2} = c_1 \frac{\sqrt{3}}{2} + c_2 \frac{1}{2}$$

$$c_1 = \frac{\sqrt{3}}{4} + \frac{c_2}{2}$$

$$0 = -c_1 \sin t + c_2 \cos t$$

$$0 = -c_1 \cdot \frac{\sqrt{3}}{2} + c_2 \frac{\sqrt{3}}{2}$$

$$3 \frac{1}{2} = c_1 \frac{\sqrt{3}}{2} + c_2 \frac{1}{2}$$

$$c_1 = \frac{3}{4}$$

$$c_2 = \frac{9}{4}$$

$$\cos(\pi/6) = 4\sqrt{3} c_1 + \frac{\sqrt{3}}{4}$$
3. \((8 + 2 = 10\text{ points})\) Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Another brine solution with concentration 2 lb/gal is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is then pumped out at a slower rate 2 gal/min.

(a) Determine a differential equation for the amount of salt \(A(t)\) in the tank at time \(t\).

\[
\frac{dA}{dt} = (2)(3) - 2 \left( \frac{A(t)}{300 + t} \right) \quad \text{(Volume: } 300 + t\text{ at time } t)\]

\[
\frac{dA}{dt} = 6 - \frac{2A}{300 + t}
\]

(b) State an initial condition for \(A(t)\).

\[A(0) = 50\]

4. \((6 + 6 = 12\text{ points})\)

(a) Solve the following differential equation for its general solution.

\[dy - (y - 1)^2 \, dx = 0\]

\[
dy = (y-1)^2 \, dx
\]

\[
\frac{1}{(y-1)^2} \, dy = dx
\]

\[- \frac{1}{y-1} = x + C
\]

\[
y - 1 = \frac{1}{C-x}
\]

The gen. solution is

\[
\begin{cases}
y = 1 + \frac{1}{C-x}, & c \text{ is arbitrary constant} \\
y = 1
\end{cases}
\]
(b) Find an explicit solution of the following initial value problem.

\[
\frac{dx}{dt} = 4(1 + x^2), \quad x(\pi/4) = 1
\]

[Hint: \( \frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2} \)]

\[
\int \frac{1}{1 + x^2} \, dx = \int 4 \, dt
\]

\[
\tan^{-1} x = 4t + C \implies x = \tan(4t + C).
\]

When \( t = \pi/4 \), \( x = 1 \): \( \tan(\pi/4) = 1 \) \( \Rightarrow C = -\pi/4 \)

\[
x = \tan(4t - \frac{\pi}{4})
\]

5. \( (8+2=10 \text{ points}) \) Consider the first-order linear nonhomogeneous differential equation

\[ x^2 y' + x(x+2)y = e^x. \]

(a) Find the general solution of the differential equation.

\[
y' + \frac{x(x+2)y}{x^2} = \frac{e^x}{x^2}, \quad \frac{x+2}{x} = 1 + \frac{2}{x} = P(x).
\]

\[
\int P(x) \, dx = \int (1 + \frac{2}{x}) \, dx = x + 2 \ln |x| \quad \text{(by standard integral)}
\]

\[
\text{I.F.} = e^{\int P(x) \, dx} = e^{x + 2 \ln |x|} = x^2 e^x.
\]

Multiplying (1) by I.F. and integrating

\[
\int x^2 e^x \, dx = \int \frac{e^x}{x^2} \, x^2 e^x \, dx = \int e^{2x} \, dx = \frac{1}{2} e^{2x} + C
\]

\[
\Rightarrow x^2 e^x y = \frac{1}{2} e^{2x} + C \implies y = c e^{-x} \left( \frac{e^x}{x^2} + \frac{e^x}{2x^2} \right).
\]

(b) Give a largest interval over which the general solution is defined.

\[
\text{tan} (\theta - \frac{3\pi}{4}) = \text{tan} (\theta + \frac{3\pi}{4})
\]

\[
= \text{tan} (\theta - \frac{3\pi}{4} + \pi) = \text{tan} (\theta + \frac{3\pi}{4})
\]

\[
(0, \infty), \quad \text{or} \quad (-\infty, 0).
\]
6. (10 points) Initially 100 milligrams of a radioactive substance was present. After 6 hours the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at time \( t \), find the amount remaining after 24 hours.

\[
\frac{dp}{dt} \propto kp \Rightarrow \frac{dp}{dt} = kp ; k < 0
\]

\[
p = p_0 e^{kt} \Rightarrow p = 100 e^{kt}
\]

\[t=6\]

\[\frac{97}{100} = 100 e^{k6} \Rightarrow 6k = \ln\left(\frac{97}{100}\right) \Rightarrow k = \frac{\ln(0.97)}{6}
\]

\[t=24\]

\[p = 100 e^{(24)k} = 100 e^{\left(\frac{\ln(0.97)}{6}\right)24} = 100 \cdot e^{\ln(0.97)^4} = 88.52
\]

7. (4+4+2=10 points) Determine whether the given set of functions are linearly independent on the interval \((-\infty, \infty)\).

(a) \(f_1(x) = x, f_2(x) = x - 1, f_3(x) = x + 3\)

\[
G_1x + G_2(x-1) + G_3(x+3) = 0
\]

\[\text{But } x = 0: \quad -c_2 + 3c_3 = 0 \Rightarrow c_2 = 3c_3
\]

\[\text{But } x = 1: \quad c_1 + 4c_2 = 0 \Rightarrow c_1 = -4c_2 \Rightarrow -12c_3 = -12c_3 \Rightarrow c_3 = 1, c_1 = -4, c_2 = 3
\]

(b) \(f_1(x) = 0, f_2(x) = x, f_3(x) = e^x\)

\[
G_1 \cdot 0 + G_2 x + G_3 e^x = 0 \text{ choose } G_1 = 1, G_2 = 0, G_3 = 0
\]

So functions are linearly dependent.

(c) \(f_1(x) = x, f_2(x) = e^x\)

\[
G_1x + G_2e^x = 0
\]

\[\text{But } x = 0 \Rightarrow c_2 = 0 \text{ This shows that functions are linearly independent.}\]
8. (10 points) Verify that the functions \( y_1 = x^3 \) and \( y_2 = x^4 \) form a fundamental set of solutions of the differential equation
\[
x^2 y'' - 6xy' + 12y = 0
\]
on the interval \((0, \infty)\).

\[
y_1' = 3x^2, \quad y_1'' = 6x \quad \Rightarrow \quad x^2 y_1'' - 6x y_1' + 12y_1 = 6x^3 - 36x^2 + 12x^3 = 0
\]
for all \( x \in (0, \infty) \).

\[
y_2' = 4x^3, \quad y_2'' = 12x^2
\]
\[
x^2 y_2'' - 6xy_2' + 12y_2 = x^2 \cdot 12x^2 - 6x \cdot 4x^3 + 12x^4 = 12x^4 - 24x^4 + 12x^4 = 0
\]

\[
W(x^3, x^4) = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = 4x^6 - 3x^6 = 3x^6 \neq 0 \text{ for } x \in (0, \infty)
\]
Functions are linearly independent.

9. (10 points) Verify that the two-parameter family of functions
\[
y = c_1 e^{2x} + c_2 e^{5x} + 6e^x
\]
is the general solution of the differential equation
\[
y'' - 7y' + 10y = 24e^x
\]
on the interval \((-\infty, \infty)\).

Let \( y_c = c_1 e^{2x} + c_2 e^{5x} = y_1 + c_2 y_2, \quad y_p = 6e^x \).

We show that \( y_1 = e^{2x}, \quad y_2 = e^{5x} \) are linearly independent solutions and \( y_p = 6e^x \) is a particular solution.

\[
y_1'' - 7y_1' + 10y_1 = 4e^{2x} - 14e^{2x} + 10e^{2x} = 0
\]
\[
y_2'' - 7y_2' + 10y_2 = 25e^{5x} - 25e^{5x} + 30e^{5x} = 0
\]

\[
\begin{vmatrix}
e^{2x} & e^{5x} \\
e^{2x} & 5e^{5x} 
\end{vmatrix} = 5e^{7x} - 2e^{7x} = 3e^{7x} \neq 0.
\]
This shows that functions are form a fundamental set of solutions. Also, \( y_p'' - 57y_p' + 10y_p = 6e^x - 42e^{2x} + 60e^x = 24e^x \). Yes
10. (10 points) If the function \( y_1(x) = xe^{-x} \) is a solution of the differential equation

\[ y'' + 2y' + y = 0, \]

find a second solution \( y_2(x) \) so that the functions \( y_1(x) \) and \( y_2(x) \) are linearly independent on the interval \((-\infty, \infty)\).

[Hint: You may use reduction of order, or a relevant formula.]

Let \( y_2 = y_1^2 \) where \( a = \int \frac{-xe^{-x} \, dx}{y_1^2} \)

\[ = \int \frac{e^{-2x}}{(xe^{-x})^2} \, dx = \int \frac{e^{-2x}}{x^2e^{-2x}} \, dx = \int x^2 \, dx = -\frac{1}{x}. \]

\[ y_2 = xe^{-x}(-\frac{1}{x}) = -e^x. \] We can just take \( y_2 = e^x. \)

*Extra-credit (2+1+2=5 points)

(a) Verify that \( y = 5 \tan(5x) \) is an explicit solution of the first-order differential equation

\[ y' = 25 + y^2. \]

\[ y' = 25 \sec^2(5x), \]

\[ 25+y^2 = 25+25\tan^2(5x) = 25 \sec^2(5x). \]

The right sides are equal at the points where they are defined.

(b) What is the domain of \( y \) as a function?

\[ x \in \left(-\frac{x}{10}, \frac{\pi}{10}\right) \cup \left(\frac{\pi}{10}, \frac{\pi}{10} + \pi n \right), \text{ where } n \text{ is an integer}. \]

(c) Give at least one interval of definition by considering \( y \) as a solution of the differential equation.

We can take \( \left(-\frac{\pi}{10}, \frac{\pi}{10}\right) \).