Abstract

Equilibrium problems are among the most interesting and intensively studied classes of problems; they include fundamental mathematical problems like optimization, variational inequalities, and complementarity problems. Many problems of practical interest in optimization, economics, and engineering involve equilibrium in their description. In recent years, these facts motivated several researchers to establish general results on the existence of equilibrium. There is a vast literature of equilibrium problems and their treatment in optimization, variational and quasivariational inequalities, and complementarity problems.

The echilibrium problems can be formulated as follows. Let V be a real topological vector space with topological dual V^*; denote the duality pairing between V and V^* by $\langle .,. \rangle$. Let K be a nonempty subset of V, and f a real bifunction defined on $K \times K$. Then, the echilibrium problem is defined as

$$\text{find } \bar{x} \in K \text{ such that } f(\bar{x}, y) \geq 0, \text{ for each } y \in K.$$

For surveys of concepts and applications, see [6], [2], [10], [9], [39], [16], [35].

The purpose of the thesis **"Echilibrium problems in engineering and economics"** is to study two particular cases of the echilibrium problem mentioned above. While the first particular case deals with a generalized hemivariational inequality, the second one treats Nash echilibrium point.

In Chapter 1 of this thesis notions and propositions of nonlinear analysis that are used in the next chapters are presented. The chapter closes with a section concerning the formulation of echilibrium problems.

Chapter 2 contains a new generalized hemivariational inequality. We begin with the presentation of the problem and give existence results on Banach spaces and on reflexive Banach spaces. The theory of this chapter is illustrated by means of applications from mathematics and engineering.

In Chapter 3 an eigenvalue problem for the generalized hemivariational inequality is treated. Again, the results are given on Banach spaces and on reflexive Banach spaces. The chapter closes with applications of the theory to engineering.

In Chapter 4 the theory of Nash echilibrium is developed. Making use of the hemivariational inequality presented earlier, we prove the existence of Nash equilibrium for a certain class of games.

Chapter 5 deals with numerical methods for computing the Nash equilibrium. The new methods developed here are based on the concept of descent direction. A numerical example is provided at the end.

Chapter 1

Some theoretical considerations

The objectives of this first chapter are to present some topics which are directly applicable to our study.

1.1 Banach spaces

This section is devoted to the presentation of some concepts concerning Banach spaces. The notions of Banach space, bidual, reflexive Banach space, norm topology, weak topology, weak* topology, $L^p(\Omega), H_0^1(\Omega)$ are recalled.

1.2 Continuity and differentiability

In this section we include some concepts and properties of usual functions, like lower/upper semicontinuity, convexity, differentiability, subdifferentiability, continuity, etc. Among the references cited we can mention the following: [54], [38], [7], [29], [56], [40].

1.3 Set-valued mappings

The third section contains some important definitions and propositions about multifunctions. All of them will be employed in the study of hemivariational inequalities. See [4], [5], [15], [28] for details on this material.

1.4 Nonsmooth analysis

The notion of generalized gradient for locally Lipschitz functions was introduced and developed by F. H. Clarke (see [20], [19], [21], [22]). Few properties are recalled here for the purpose of this work.

1.5 Duality mapping

We consider certain properties of duality mapping and semi-inner products $(.,.)_{\pm}$, which will be involved in the study of eigenvalue problem for hemivariational inequalities. For comprehensive material we refer to [27], [44], [55].

1.6 Echilibrium problems

The aim of this section is to present an overview on echilibrium problems and on its particular cases. One of our purposes is to reveal the connection between this theory and the present thesis. For more details about echilibrium problems and its applications, one may see [6], [2], [10], [9], [39], [16], [35].

Chapter 2

Hemivariational inequalities

2.1 Echilibrium in engineering and solution of a hemivariational inequality

The echilibrium problems from engineering give rise to the following important inequalities: variational inequalities, which, with a research "life" of some thirty years now, is mainly concerned with convex energy functions, and hemivariational inequalities, which is more recent and is concerned with nonconvex energy functions.

The mathematical theory of hemivariational inequalities and their applications in mechanics, engineering or economics, were introduced and developed by P.D. Panagiotopoulos ([46], [47], [48], [49], [50], [51]). This theory has been developed in order to fill the gap existing in the variational formulations of boundary value problems (B.V.P.s) when nonsmooth and generally nonconvex energy functions are involved in the formulations of the problem. In fact, this theory of hemivariational inequalities may be considered as an extension of the theory of variational inequalities ([30], [32], [42], [37]). For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs ([47], [51], [45], [44]).

The purpose of the present work is to extend these results in the framework of hemivariational inequalities governed by two variable operators.

2.2 A generalized hemivariational inequality

Let V be a real Banach space endowed with the norm topology, and let V^* be its dual endowed with the weak*-topology. Throughout the present thesis the duality pairing between a Banach space and its dual is denoted by $< .,. >$. We assume that the following statements are valid:

(H1) $C \subseteq V$ is a nonempty convex subset of V;

(H2) $T : V \to L^p\left(\Omega, \Re^k\right)$ is a linear and continuous operator, where $1 \leq p < \infty, k \geq 1$ and $\Omega \subseteq \Re^n$ is a bounded open set in n-dimensional Euclidean space;

(H3) $A : C \times C \rightsquigarrow V^*$ is a set-valued mapping;

(H4) $j = j(x, y) : \Omega \times \Re^k \to \Re$ is a Caratheodory function, which is locally Lipschitz with respect to the second variable and satisfies the following condition

$$\begin{aligned} \exists h_1 &\in L^{\frac{p}{p-1}}(\Omega, \Re), \exists h_2 \in L^{\infty}(\Omega, \Re) \text{ such that} \\ |z| &\leq h_1(x) + h_2(x)|y|^{p-1} \text{ a.e. } x \in \Omega, \forall y \in \Re^k, \forall z \in \partial j(x, y) \end{aligned}$$

$$\text{where, } j^0(x, y)(h) = \limsup_{\substack{y' \to y \\ t \to 0^+}} \frac{j(x, y' + th) - j(x, y')}{t}$$

is the Clarke derivative of the locally Lipschitz mapping $j(x, .), x \in \Omega$ fixed, at the point $y \in \Re^k$ with respect to the direction $h \in \Re^k$,

$$\text{and, } \partial j(x, y) = \left\{ z \in \Re^k : \langle z, h \rangle \leq j^0(x, y)(h), \forall h \in \Re^k \right\}$$

is the Clarke generalized gradient of the mapping $j(x, .)$ at the point $y \in \Re^k$.

Using the above notation, the problem (**A.-M. Croicu**) to be solved becomes:

Find $u \in C$ such that

$$\sup_{f\in A(u,u)}\langle f,v-u\rangle+\int_{\Omega}j^{0}\left(x,Tu\left(x\right)\right)\left(Tv\left(x\right)-Tu\left(x\right)\right)dx\geq0,\forall v\in C. \quad \text{(P)}$$

Remark 2.2.1 *A very important case is when the Banach space V is $H_0^1(\Omega)$ and the operator $A(u,v)$ is a nonlinear elliptic differential operator. Generally, this kind of operator $A(.,.)$ is* `monotone` *only with respect to v, i.e. with respect to the higher order term (usually, the gradient), where u is fixed. In other words, these two variables u and v do not play the same role with respect to the operator A properties. Seeking the solution along the 'diagonal' (u,u) is motivated by the fact that the modelled phenomena depend on the unknown function u, as well as on the gradient of unknown function u.*

2.3 The generalized hemivariational inequality on Banach spaces

2.3.1 Existence results

Definition 2.3.1 *We say that the set-valued mapping $A(.,v): C \rightsquigarrow V^*, v\in C$ fixed,* ***has the monotone property (M)*** *if it verifies the relation*

$$\sup_{f\in A(u,v)}\langle f,u-v\rangle\geq\sup_{g\in A(v,v)}\langle g,u-v\rangle,\forall u\in C. \quad (M)$$

Theorem 2.3.2 ***(A.-M. Croicu, [26]):*** *Let V be a real Banach space endowed with the norm topology and let V^* be its dual endowed with the weak*-topology. Assume that all the hypotheses (H1)-(H4) are satisfied. Moreover, the following assumptions hold:*

(i) for each $v\in C$, the set-valued mapping $A(.,v): C\rightsquigarrow V^*$ has the monotone property (M) and it is weakly-upper semicontinuous from the line segments of C in V^*;*

(ii) for each $u\in C$, the set-valued mapping $A(u,.): C\rightsquigarrow V^*$ is weakly-upper semicontinuous;*

(iii) there exists a compact subset $K \subseteq C$, and an element $u_0 \in C$ such that the coercivity condition

$$\sup_{f \in A(u,u)} \langle f, u_0 - u \rangle + \int_\Omega j^0 \left(x, Tu\left(x\right)\right) \left(Tu_0\left(x\right) - Tu\left(x\right)\right) dx < 0, \forall u \in C \diagdown K$$

holds;

(iv) for each $u, v \in C$, the set $A\left(u, v\right)$ is weakly-compact.*

Then the problem (P) admits a solution $u \in C$.

If in addition $A\left(u, u\right)$ is a convex set, then u is also a solution of the following problem:

$$\text{Find } u \in C, f \in A\left(u, u\right) \text{ such that}$$

$$\langle f, v - u \rangle + \int_\Omega j^0 \left(x, Tu\left(x\right)\right) \left(Tv\left(x\right) - Tu\left(x\right)\right) dx \geq 0, \forall v \in V. \qquad (Pc)$$

Theorem 2.3.3 ***(A.-M. Croicu, [26]):*** *Let V be a real Banach space endowed with the norm topology and let V^* be its dual endowed with the weak*-topology. Assume that $C \subseteq V$ is closed and all the hypotheses (H1)-(H4) are satisfied. Moreover, the following conditions hold:*

(i) for each $v \in C$, the set-valued mapping $A\left(., v\right) : C \rightsquigarrow V^*$ has the monotone property (M) and it is weakly-upper semicontinuous from the line segments of C in V^*;*

(ii) whenever D is a convex subset of C and $\left(v_i\right)_{i \in I}$ is a net in C converging to the element $v \in D$, then

$$\sup_{g \in A(z, v_i)} \langle g, z - v_i \rangle + \int_\Omega j^0 \left(x, Tv_i\left(x\right)\right) \left(Tz\left(x\right) - Tv_i\left(x\right)\right) dx \geq 0, \forall z \in D$$

implies

$$\sup_{g \in A(z, v)} \langle g, z - v \rangle + \int_\Omega j^0 \left(x, Tv\left(x\right)\right) \left(Tz\left(x\right) - Tv\left(x\right)\right) dx \geq 0, \forall z \in D;$$

(iii) there exists a compact subset $K \subseteq C$, and an element $u_0 \in C$ such that the coercivity condition

$$\sup_{f \in A(u,u)} \langle f, u_0 - u \rangle + \int_{\Omega} j^0 \left(x, Tu\left(x\right)\right) \left(Tu_0\left(x\right) - Tu\left(x\right)\right) dx < 0, \forall u \in C \diagdown K$$

holds;

(iv) for each $u \in C$, the set $A(u,u)$ is weakly-compact;*

(v) for each finite dimensional subspace Y of V, the set-valued mapping $A : C \times C \rightsquigarrow V^*$ is weakly-upper semicontinuous on the diagonal of $(C \cap Y) \times (C \cap Y)$.*

Then the problem (P) admits a solution $u \in C$.

If in addition $A(u,u)$ is a convex set, then u is also solution of the problem (Pc).

Remark 2.3.4 *The coercivity condition (iii) which appears in the Theorems 2.3.2 or 2.3.3 tells us that we have to look for solutions of hemivariational inequality (P) in the compact set K.*

2.3.2 Applications

This section is dedicated to applications of Theorems 2.3.2 and 2.3.3 in mathematics and engineering. First, there are provided alternative results to those presented in [52], [16]. Second, an application to engineering is studied.

Example 2.3.5 ***(A.-M. Croicu, [26]):*** *Let us analyze a very general situation which leads us to the hemivariational inequality problem (P). For instance, let us consider an open, bounded, connected subset $\Omega \subseteq \Re^3$ referred to a fixed Cartesian coordinate system $Ox_1x_2x_3$ and we formulate the problem : find the function u that satisfies*

$$-\Delta u + h(u) = g \text{ in } \Omega \tag{2.1}$$

$$u = 0 \text{ on } \Gamma. \tag{2.2}$$

Here Γ is the boundary of Ω and we assume that Γ is sufficiently smooth ($C^{1,1}$-boundary is sufficient) and h is a continuous function. Moreover, u may represent the temperature in the case of heat conduction problems, whereas in problems of hydraulics and electrostatics the pressure and the electric potential are represented, respectively. See for instance [31] for a comprehensive material about mathematical modelling.

We seek a function u such that to verify (2.1), (2.2) with

$$-g \in \partial j\left(x, u\right) \tag{2.3}$$

where the function g is known, $j : \Omega \times \Re \to \Re$ satisfies the assumption (H4), and $\partial j\left(x, y\right)$ denotes the Clarke generalized gradient of the mapping $j\left(x, .\right)$ at the point $y \in \Re, x \in \Omega$ fixed.

Let us consider the Sobolev space $V = H_0^1\left(\Omega\right)$. We may ask in addition that u is constrained to belong to a compact convex set $C \subseteq V$ due to some technical reasons, e.g., constraints for the temperature or the pressure of the fluid, etc. Let us note that there exist a linear monotone continuous operator $B : C \to V^*$ and a continuous operator $D : C \to V^*$ such that

$$\left\langle B\left(u\right), v\right\rangle = \int_\Omega \nabla u \nabla v dx, \forall u, v \in V,$$

$$\left\langle D\left(u\right), v\right\rangle = \int_\Omega uv dx, \forall u, v \in V.$$

Thus, if we consider the following multivalued mapping

$$\begin{aligned} A &: C \times C \rightsquigarrow V^* \\ A\left(u, v\right) &= B\left(u\right) + D\left(h\left(v\right)\right) \end{aligned}$$

then we are lead to the following problem: find $u \in C$ such that for any $v \in C$

$$\sup_{f \in A(u,u)} \left\langle f, v - u\right\rangle + \int_\Omega j^0\left(x, u\right)\left(v - u\right) dx \geq 0. \qquad \textit{(Peng)}$$

Since the multivalued operator A satisfies the assumptions (i), (ii), (iii), (iv) of the Theorem 2.3.2 and the embedding of V in $L^2\left(\Omega\right)$ is linear and continuous, we can prove the existence of solutions of (Peng) by simply applying this theorem.

2.4 The generalized hemivariational inequality on reflexive Banach spaces

2.4.1 Existence results

Theorem 2.4.1 ***(A.-M. Croicu, [24]):*** *Let V be a real reflexive Banach space endowed with the norm topology and let V^* be its dual endowed with the weak*-topology.*

Assume that the hypotheses (H2)-(H4) are satisfied and $C \subseteq V$ is a nonempty bounded closed convex subset of V. Moreover, the following assumptions hold:

(i) for each $v \in C$, the set-valued mapping $A(.,v) : C \rightsquigarrow V^*$ is weakly-upper semicontinuous from the line of C into V^*, concave and monotone;

(ii) for each $u \in C$, the set-valued mapping $A(u,.) : C \rightsquigarrow V^*$ is weakly-upper semicontinuous;

(iii) for each $u, v \in C$, the set $A(u,v)$ is weakly-compact.

Then the problem (P) admits a solution.

If in addition $A(u,u)$ is a convex set, then u is also solution of the following problem:

$$\text{Find } u \in C, f \in A(u,u) \text{ such that}$$

$$\langle f, v-u \rangle + \int_\Omega j^0(x, Tu(x))(Tv(x) - Tu(x))\,dx \geq 0, \forall v \in V. \qquad (Pc)$$

In the Theorem 2.4.1, the subset C was bounded. In order to prove a similar result when C is an unbounded set, we refer to the so-called 'recession analysis' (see [1]).

Let us consider a nonempty closed convex subset C of a real reflexive Banach space V.

A vector y is called a **recession direction in C corresponding to the vector x** if

$$\forall t > 0, x + ty \in C.$$

Recession directions are independent of x and they determine a closed convex cone called the **recession cone of C**:

$$C_\infty := \cap_{t>0}\left[\frac{C-u_0}{t}\right] \text{ , where } u_0 \in C \text{ is an arbitrarly chosen element.}$$

We define the set $R(A,j,C)$ of **asymptotic directions** by

$$R\left(A,j,C\right)=\left\{\begin{array}{c} w\in C_\infty \text{ s.t. } \exists\left(u_n\right)\subseteq C, t_n:=\|u_n\|\rightarrow\infty, w_n:=\frac{u_n}{\|u_n\|}\rightharpoonup w, \\ \\ \inf_{f\in A(u_n,u_n)}\langle f,u_n\rangle-\int_\Omega j^0\left(x,Tu_n\left(x\right)\right)\left(-Tu_n\left(x\right)\right)dx\leq 0 \end{array}\right\}$$

Theorem 2.4.2 ***(A.-M. Croicu, [24]):*** *Let V be a real reflexive Banach space endowed with the norm topology and let V^* be its dual endowed with the weak*-topology.*

Assume that all the hypotheses (H2)-(H4) are satisfied and $C\subset V$ is a nonempty unbounded closed convex subset of V such that $0\in C$. Moreover,

(i) for each $v\in C$, the set-valued mapping $A\left(.,v\right):C\rightsquigarrow V^*$ is weakly-upper semicontinuous from the line segments of C into V^*, concave and monotone;

(ii) for each $u\in C$, the set-valued mapping $A\left(u,.\right):C\rightsquigarrow V^*$ is weakly-upper semicontinuous;

(iii) $R\left(A,j,C\right)=\emptyset$;

(iv) for each $u,v\in C$, the set $A\left(u,v\right)$ is weakly-compact.

Then the problem (P) admits a solution.

If in addition the set $A(u,u)$ is convex, then the problem (Pc) admits solution also.

2.4.2 Applications

The theorems 2.4.1 and 2.4.2 are considered in the framework of [52], [17], and similar results are deduced. Moreover, Brouwer's fixed theorem is proved by the means of theorem 2.4.1. Finally, the same example 2.3.5 is analyzed on the reflexive Banach space $H_0^1\left(\Omega\right)$.

Chapter 3

Eigenvalue problems

3.1 Formulation of the problem

The study of eigenvalue problems for hemivariational inequalities has a deep practical motivation. For instance, the loading-unloading problems and thus also the hysteresis problems are typical examples for the theory of hemivariational inequalities and can be reduced to the study of the eigenvalue problem. Indeed, D. Motreanu and P. D. Panagiotopoulos ([51], [43]) proved that the global behavior of a loading-unloading problem of a deformable body is governed by a sequence of hemivariational inequality expressions, one for each branch. They proved that the changing of branch leads to an eigenvalue problem. The stability of a Von Karman plate in adhesive contact with a rigid support or of Von Karman plates adhesively connected in sandwich form is another motivation for the study of eigenvalue problems for hemivariational inequalities ([33], [34]). Recent papers deal with eigenvalue hemivariational inequalities on a sphere-like type manifold ([11], [12]), with nonsymmetric perturbed eigenvalue hemivariational inequalities ([18], [53]), which imply useful applications in adhesively connected plates, etc.

Our goal is to study the following eigenvalue problem **(A.-M. Croicu,**

[25]):

$$\text{Find } u \in V, \lambda \in \Re \setminus \{0\} \text{ such that}$$

$$\sup_{f \in A(u,u)} \langle f, v-u \rangle + \int_\Omega j^0 (x, Tu(x)) (Tv(x) - Tu(x))\, dx \geq \lambda (v-u, u)_+, \forall v \in V \tag{EP}$$

where, $(.,.)_+$ is the semi-inner product on Banach space V.

3.2 Existence results for the eigenvalue problem

Let V be a real Banach space endowed with the norm topology and let V^* be its dual endowed with the weak*-topology.

Theorem 3.2.1 ***(A.-M. Croicu, [25]):*** *Assume that all the hypotheses (H2)-(H4) are satisfied. Moreover, the following assumptions hold:*

(i) for each $v \in V$, the set-valued mapping $A(.,v) : V \rightsquigarrow V^*$ has the monotone property (M) and it is weakly-upper semicontinuous from the line segments of V in V^* ;*

(ii) for each $u \in V$, the set-valued mapping $A(u,.) : V \rightsquigarrow V^*$ is weakly-upper semicontinuous;*

(iii) there exists a compact subset $K \subseteq V$, and an element $u_0 \in V$ such that

$$\|u_0\| \leq \|u\| \ , \forall u \in V \backslash K,$$

$$\sup_{f \in A(u,u)} \langle f, u_0 - u \rangle + \int_\Omega j^0 (x, Tu(x)) (Tu_0(x) - Tu(x))\, dx < 0, \forall u \in V \diagdown K;$$

(iv) for each $u, v \in V$, the set $A(u,v)$ is weakly-compact.*

Then for every $\lambda < 0$, the problem (EP) admits a solution $u \in V$.

If in addition $A(u,u)$ is a convex set, then the following problem (EPc):

$$\textit{Find } u \in V, \lambda \in \Re \setminus \{0\}, f \in A(u,u) \textit{ such that}$$

$$\langle f, v-u \rangle + \int_\Omega j^0 (x, Tu(x)) (Tv(x) - Tu(x))\, dx \geq \lambda (v-u, u)_+, \forall v \in V \tag{EPc}$$

admits a solution $u \in V, f \in A(u,u)$ for every $\lambda < 0$.

Theorem 3.2.2 ***(A.-M. Croicu, [25]):*** *Assume that all the hypotheses (H2)-(H4) are satisfied, and V is a real reflexive Banach space. Moreover,*

(i) for each $v \in V$, the set-valued mapping $A(.,v) : V \rightsquigarrow V^*$ is weakly-upper semicontinuous from the line segments of V into V^* , concave and monotone;

(ii) for each $u \in V$, the set-valued mapping $A(u,.) : V \rightsquigarrow V^*$ is weakly-upper semicontinuous;

(iii)$R(A, j, V) = \emptyset$;

(iv) for each $u, v \in V$, the set $A(u,v)$ is weakly-compact.

Then the problem (EP) admits a solution.

If in addition the set A(u,u) is convex, then the problem (EPc) admits solution also.

Remark 3.2.3 *Under the assumptions of the Theorems 3.2.1, 3.2.2 not only the eigenvalue problem (EP) but also the hemivariational inequality (P) admits solution.*

3.3 Applications

Example 3.3.1 ***(A.-M. Croicu, [25]):*** *Our results can be applied directly to the study of B. V. P.s in Engineering. Let us analyze a very general situation which leads us to the hemivariational inequality problem (EP). For instance, let us consider an open, bounded, connected subset $\Omega \subseteq \Re^3$ referred to a fixed Cartesian coordinate system $Ox_1x_2x_3$ and we formulate the problem : find the function u that satisfies*

$$-\Delta u + h(u) + cu = g \text{ in } \Omega \tag{3.1}$$

$$u = 0 \text{ on } \Gamma. \tag{3.2}$$

Here Γ is the boundary of Ω and we assume that Γ is sufficiently smooth ($C^{1,1}$-boundary is sufficient), c is a given constant, and h is a continuous function, which has the property

$$u(x)\,h(u(x)) \geq 0, \forall x \in \Omega. \tag{3.3}$$

In order to physically motivate problem (3.1),(3.2) in a simple way, we interpret u as the temperature of a medium in a region Ω. The differential equation in (3.1) describes a stationary temperature state with the heat source $f - h(u) - cu$ that depends on temperature (see [58]). We seek a function u such that to verify (3.1), (3.2) with

$$-g \in \partial j\left(x, u\right) \tag{3.4}$$

where $j\left(x, .\right)$ is a locally Lipschitz function. Let us consider the Sobolev space $V = H_0^1\left(\Omega\right)$, which can be viewed as a Hilbert space endowed with the inner-product

$$\left(u, v\right) = \int_\Omega uvdx,\ \forall u, v \in V.$$

If we consider some further assumptions, the linear monotone continuous operator $B : V \to V^*$,

$$\left\langle B\left(u\right), v\right\rangle = \int_\Omega \nabla u \nabla v dx, \forall u, v \in V,$$

the duality isomorphism $\Im : V \to V^*$,

$$\left\langle \Im u, v\right\rangle = \left(u, v\right),\ \forall u, v \in V,$$

and the following multivalued mapping

$$\begin{array}{rcl} A & : & V \times V \rightsquigarrow V^* \\ A\left(u, v\right) & = & B\left(u\right) + \Im\left(h\left(v\right)\right), \end{array}$$

we are led to the following problem

$$\textit{find } u \in V \textit{ such that for any } v \in V$$

$$\sup_{f \in A(u,u)} \left\langle f, v - u\right\rangle + \int_\Omega j^0\left(x, u\right)\left(v - u\right) dx \geq \left(-c\right)\left(v - u, u\right)_+. \qquad (EPeng)$$

Since all the assumptions of the Theorem 3.2.2 are ensured and the embedding $V \subseteq L^2\left(\Omega\right)$ is linear and continuous, we can prove the existence of solutions of (EPeng) for all $c > 0$.

Chapter 4

Nash equilibrium

4.1 Echilibrium in economics and Nash equilibrium

Traditional game theory takes as its basic distinction that between cooperative games and noncooperative games. In cooperative games, the players are assumed to be free to communicate in any way they choose before and during the game. More importantly, they are also assumed to be able to bind themselves to any agreements that may be reached during such bull sessions. A noncooperative game ought properly to be defined as a game that is not cooperative. More often, the terminology is used to signify a game in which agreements are never binding on the players.

The notion of Nash equilibrium of an n-person noncooperative game has an important significance in game theory and economic applications.

4.2 Nash equilibrium

Let us consider X, Y two Hausdorff topological vector spaces, as well as the subsets $K_1 \subseteq X$, $K_2 \subseteq Y$.

Consider a non-cooperative game of two players: the first players task is to choose a strategy x from K_1, and the second players task is to choose a strategy y from K_2. The traditional model for the game theory classifies the strategies of each player using a loss function (payoff function). Denote by $f : K_1 \times K_2 \to \Re$ the loss function associated with the first player and by $g : K_1 \times K_2 \to \Re$ the loss function associated with the second player. It is obvious that each player makes decisions in order to minimize his loss. For this reason, we are led to the following definition:

Definition 4.2.1 *The point $(a, b) \in K_1 \times K_2$ is called **a Nash equilibrium** point for the game if:*

$$\begin{array}{l} f(a,b) \leq f(x,b) \ \ , \forall x \in K_1 \\ g(a,b) \leq g(a,y) \ \ , \forall y \in K_2 \end{array} .$$

These inequalities assert that the strategies (a, b) are optimal responses of each player, if the strategy of the partner is unchanged ([3]).

In [36] there are presented other concepts which are close related to the Nash equilibrium points. They are weak and strong Nash stationary points. These notions will be very useful in the next chapter, when numerical methods for finding Nash equilibrium are considered.

4.3 Existence result

This section provides an existence result for Nash equilibrium, derived from the aforementioned theorem 2.3.2.

Theorem 4.3.1 ***(A.-M. Croicu):*** *Consider $K_1 \subseteq \Re^{n_1}, K_2 \subseteq \Re^{n_2}$ nonempty compact convex subsets, $f, g : K_1 \times K_2 \to \Re$ continuous functions s.t.*

$$\begin{array}{l} \text{for every } y \in K_2 \text{ fixed, the mapping } x \in K_1 \longmapsto f(x,y) \text{ is strictly convex} \\ \text{for every } x \in K_1 \text{ fixed, the mapping } y \in K_2 \longmapsto g(x,y) \text{ is strictly convex} \end{array} .$$

Then at least one Nash equilibrium exists.

Chapter 5

The computation of Nash equilibrium

5.1 Some numerical algorithms

Practical algorithms for finding Nash equilibrium have been elaborated only recently. These numerical algorithms seem to be important in real-word applications, where the players do not know each other's objective functions and other relevant information ([13]). The players only have their own tentative decisions to communicate to each other during each phase of the computation, as seen in [8] and [14].

This section is devoted to the presentation of some numerical methods found in literature. We emphasize on "parallel decision making" ([41]), "inaccurate search algorithm" ([41]), "parallel gradient descent" ([13]) and on "relaxation algorithm" ([8], [57]).

5.2 A gradient-type and relaxation-type method

In this section, we consider a class of games and we investigate two numerical methods for computing the Nash equilibrium. The numerical methods presented here start from the ideas of Nash stationary points ([36]) and computation of Nash equilibria via parallel gradient descent ([13]).

5.2.1 Directions of descent

The following proposition will be employed in the numerical methods presented in this thesis.

Proposition 5.2.1 ***(A.-M. Croicu, [23]):*** *Let $C_1 \subset \Re^{n_1}$, $C_2 \subset \Re^{n_2}$ be nonempty compact subsets, let $f, g : C_1 \times C_2 \to \Re$ be differentiable functions, let $x \in intC_1$, $y \in intC_2$, $u \in \Re^{n_1}$, $v \in \Re^{n_2}$ be arbitrary elements. If*

$$\left\langle u, \frac{\partial f}{\partial x}(x,y) \right\rangle < 0 \qquad \text{and} \qquad \left\langle v, \frac{\partial g}{\partial y}(x,y) \right\rangle < 0 \tag{5.1}$$

then there exists $a > 0$, $b > 0$ such that

$$\begin{array}{l} x + tu \in C_1 \ , \ f(x+tu, y) < f(x,y) \ \ , \forall t \in (0,a] \\ y + tv \in C_2 \ , \ g(x, y+tv) < g(x,y) \ \ , \forall t \in (0,b] \end{array} .$$

Example 5.2.2 *If $\frac{\partial f}{\partial x}(x,y) \neq 0$ and $\frac{\partial g}{\partial y}(x,y) \neq 0$ then $u = -\frac{\partial f}{\partial x}(x,y)$ and $v = -\frac{\partial g}{\partial y}(x,y)$ are good candidates as descent directions satisfying (5.1).*

Example 5.2.3 *If there exists $i \in \{1, 2, ..., n_1\}$ such that $\frac{\partial f}{\partial x_i}(x,y) \neq 0$ and $j \in \{1, 2, ..., n_2\}$ such that $\frac{\partial g}{\partial y_j}(x,y) \neq 0$ then $u = -sgn \frac{\partial f}{\partial x_i}(x,y) e^i$ and $v = -sgn \frac{\partial g}{\partial y_j}(x,y) e^j$ are good candidates as descent directions, as well.*

Recall that e^k is the vector with all components equal to zero, except the component on the position 'k', which is one.

We consider the following **problem (PEN):** *Find the Nash equilibrium for the two-person noncooperative game characterized by the following:*

$K_1 \subset \Re, K_2 \subset \Re$ are nonempty compact convex subsets

$$K_1 = [m_1, M_1] \ , K_2 = [m_2, M_2]$$

$f, g : K_1 \times K_2 \to \Re$ *are continuous differentiable functions ,with* $f \in \mathcal{L}_1(K_1 \times K_2)$, $g \in \mathcal{L}_2(K_1 \times K_2)$ (see [36] for notation) *and*

$$\begin{array}{l} \forall y \in K_2 \text{ fixed, the aplication } x \in D_1 \longmapsto f(x,y) \text{ is strictly convex} \\ \forall x \in K_1 \text{ fixed, the aplication } y \in D_2 \longmapsto g(x,y) \text{ is strictly convex} \end{array} .$$

According to the Theorem 4.3.1, the problem (PEN) admits a Nash equilibrium point at least. Our goal is to elaborate numerical algorithms which supplies these points. Taking into account the Examples 5.2.2 and 5.2.3, we can elaborate two numerical algorithms, which generate a sequence $(x^p, y^p)_{p \in N^*}$ converging to a Nash equilibrium point.

5.2.2 A gradient-type algorithm

THE GRADIENT TYPE ALGORITHM **(A.-M. Croicu, [23]):**

1. Choose $(x^1, y^1) \in (int\,[m_1, M_1]) \times (int\,[m_2, M_2])$ *, put* $p = 1$ *;*

2. Compute $\frac{\partial f}{\partial x}(x^p, y^p)$ *and* $\frac{\partial g}{\partial y}(x^p, y^p)$.

If $\left[\frac{\partial f}{\partial x}(x^p, y^p) = 0 \text{ and } \frac{\partial g}{\partial y}(x^p, y^p) = 0\right]$ *put* $x^q = x^p$, $y^q = y^p, \forall q > p$ *and stop the algorithm;*

3. If $\frac{\partial f}{\partial x}(x^p, y^p) \neq 0$ *then*

set $a_p := \left(1 - \frac{1}{2^p}\right) \sup\{\alpha > 0 : x^p - \alpha \frac{\partial f}{\partial x}(x^p, y^p) \in [m_1, M_1],$

$\forall \hat{\alpha} \in (0, \alpha], f\left(x^p - \hat{\alpha} \frac{\partial f}{\partial x}(x^p, y^p), y^p\right) < f(x^p, y^p)\}$

else set $a_p := 1$ *;*

4. If $\frac{\partial g}{\partial y}(x^p, y^p) \neq 0$ *then*

set $b_p := \left(1 - \frac{1}{2^p}\right) \sup\{\beta > 0 : y^p - \beta \frac{\partial g}{\partial y}(x^p, y^p) \in [m_2, M_2],$

$\forall \hat{\beta} \in (0, \beta], g\left(x^p, y^p - \hat{\beta} \frac{\partial g}{\partial y}(x^p, y^p)\right) < g(x^p, y^p)\}$

else set $b_p := 1$ *;*

5. Let $\left\{ \begin{array}{l} x^{p+1} := x^p - a_p \frac{\partial f}{\partial x}(x^p, y^p) \\ y^{p+1} := y^p - b_p \frac{\partial g}{\partial y}(x^p, y^p) \end{array} \right.$ *;*

6. Increase p:=p+1 and go to the second step .

Consider now $(x^p, y^p)_{p\in N^*}$ the sequence obtained by the gradient type algorithm. Because the sequence $(x^p, y^p)_{p\in N^*}$ is bounded, Cesaro's Lemma implies that there exists a subsequence, denoted the same, converging to a certain point $(a, b) \in K_1 \times K_2$.

Theorem 5.2.4 *The limit point $(a, b) \in K_1 \times K_2$, obtained by the gradient-type algorithm, is a Nash equilibrium point of the problem (PEN).*

5.2.3 A relaxation-type algorithm

THE RELAXATION TYPE ALGORITHM **(A.-M. Croicu, [23]):**

1. Choose $\left(x^1, y^1\right) \in (int\,[m_1, M_1]) \times (int\,[m_2, M_2])$ *, put* $p = 1$ *;*

2. Compute $\frac{\partial f}{\partial x}(x^p, y^p)$ *and* $\frac{\partial g}{\partial y}(x^p, y^p)$ *.*

If $\left[\frac{\partial f}{\partial x}(x^p, y^p) = 0 \text{ and } \frac{\partial g}{\partial y}(x^p, y^p) = 0\right]$ *put* $x^q = x^p,\ y^q = y^p, \forall q > p$ *and stop the algorithm;*

3. If $\frac{\partial f}{\partial x}(x^p, y^p) \neq 0$ *then*

$$\text{set } a_p := \left(1 - \tfrac{1}{2^p}\right)\sup\{\alpha > 0 : x^p - \alpha u \in [m_1, M_1],\ \forall \hat{\alpha} \in (0, \alpha], f(x^p - \hat{\alpha} u, y^p) < f(x^p, y^p)\}$$

$$\text{where, } u = sgn\frac{\partial f}{\partial x}(x^p, y^p)$$

else set $a_p := 1$ *;*

4. If $\frac{\partial g}{\partial y}(x^p, y^p) \neq 0$ *then*

$$\text{set } b_p := \left(1 - \tfrac{1}{2^p}\right)\sup\{\beta > 0 : y^p - \beta v \in [m_2, M_2],\ \forall \hat{\beta} \in (0, \beta],\ g\left(x^p, y^p - \hat{\beta} v\right) < g(x^p, y^p)\}$$

$$\text{where, } v = sgn\frac{\partial g}{\partial y}(x^p, y^p)$$

else set $b_p := 1$ *;*

5. Let $\left\{ \begin{array}{l} x^{p+1} := x^p - a_p sgn\frac{\partial f}{\partial x}(x^p, y^p) \\ y^{p+1} := y^p - b_p sgn\frac{\partial g}{\partial y}(x^p, y^p) \end{array} \right.$ *;*

6. Increase p:=p+1 and go to the second step .

Using the same argument as in the gradient-type method, we note that there exists a subsequence generated by the relaxation-type algorithm, which is convergent to a certain point $(a, b) \in K_1 \times K_2$.

Theorem 5.2.5 *The limit point* $(a, b) \in K_1 \times K_2$*, obtained by the relaxation-type algorithm, is a Nash equilibrium point of the problem (PEN).*

5.3 Numerical example

Let us consider the following 2-player noncooperative game example.

$$f : [0, 2] \times [1, 3] \to \Re,\ f(x, y) = 2x^2 - 2xy + 5y^2 - 6x - 6y$$

$$g : [0, 2] \times [1, 3] \to \Re,\ g(x, y) = x^2 + xy + y^2 - 3x - 6y.$$

The Nash equilibrium point is computed exactly to be $N(2, 2)$. Next, the Nash equilibria is computed numerically. The results of the computational process are given below.

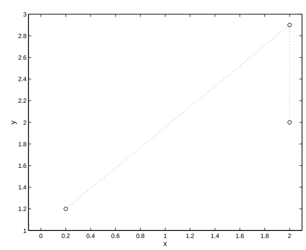

Figure 5.1: **Parallel decision making**

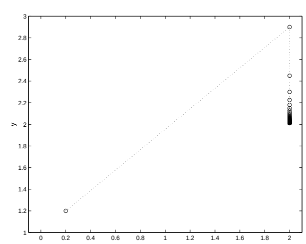

Figure 5.2: **Relaxation algorithm**

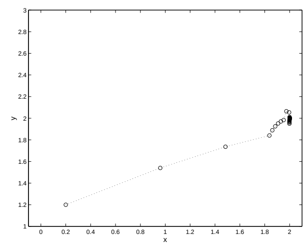

Figure 5.3: **Gradient-type algorithm**

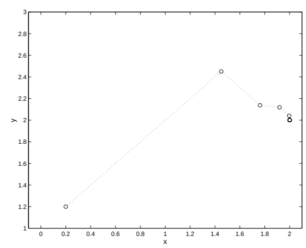

Figure 5.4: **Relaxation-type algorithm**

We have developed two new numerical methods for computation of Nash equilibria. As the numerical example shows, the results are quite satisfactory. Plus, we can underline some remarks. First, the gradient-type and relaxation-type algorithms do not require much information, only the expressions of the functions and their gradients. Second, because the stepsize search may be simplified, as we did in our example, the implementation of these numerical methods is quite easy. We may conclude that the two numerical methods presented here can be successfully applied to all noncooperative games.

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