1. (10 points) Find the general solution of the following differential equation.

\[ y'' - 4y' + 5y = 0 \]

\[ \text{A. E.} \quad m^2 - 4m + 5 = 0 \quad \Rightarrow \quad m = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} \]

\[ = \frac{4 \pm 2i}{2} = 2 \pm i \]

The general solution is \[ y = e^{2x}(c_1 \cos x + c_2 \sin x) \], where \( c_1 \) and \( c_2 \) are arbitrary constants.

2. (10 points) Solve the following initial-value problem.

\[ \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} - 5y = 0, \quad y(1) = 0, \ y'(1) = 2 \]

\[ \text{A. E.} \quad m^2 - 4m + 5 = 0 \]

\[ (m + 1)(m - 5) = 0 \quad \Rightarrow \quad m = -1, 5 \]

The general solution is \[ y = c_1 e^{-x} + c_2 e^{5x} \], where \( c_1, c_2 \) are arbitrary constants.

\[ y(1) = 0 \implies 0 = c_1 e^{-1} + c_2 e^{5} \implies -c_1 e^{-1} = c_2 e^{5} \]

\[ c_2 = -\frac{c_1}{e^{6}} \cdot \]

\[ y' = -c_1 e^{-x} + 5c_2 e^{5x} \]

\[ y'(1) = 2 \implies 2 = -c_1 e^{-1} + 5c_2 e^{5} = -\frac{c_1}{e} - 5e^{5} \cdot \frac{c_1}{e^{6}} = -\frac{c_1}{e} - \frac{5c_1}{e^{6}} \]

\[ 2e = -6c_1 \implies c_1 = -\frac{e}{3} \cdot \quad c_2 = -\frac{e}{3} \cdot \quad c_1 = +\frac{e}{2} \cdot \quad \]

\[ \frac{1}{e^{6}} = +\frac{1}{3e^{5}} \]

The solution is \[ y = -\frac{e}{3} e^{-x} - \frac{1}{e^{5}} e^{5x} = -\frac{1}{2} e^{-x} + \frac{1}{2} e^{5(x-1)} \]
3. (10 points) Find the general solution of the following differential equation by the method of undetermined coefficients.

\[ y'' + y' - 6y = 2x. \]

Let \[ y' = A + B \]

\[ y' = 2^x + c_2 e^{3x}. \]

Let \[ y = A + B \]

Then \[ y'' = 0, \quad y' = A, \quad \text{and} \quad 0 + A - 6(A + B) = 2x \]

\[ A - 6B = 6Ax = 2x \Rightarrow -6A = 2 \]

\[ A - 6B = 0 \Rightarrow B = \frac{1}{6} (-\frac{1}{3}) = -\frac{1}{18} \]

\[ y = c_1 e^{2x} + c_2 e^{-3x} - \frac{1}{3} x - \frac{1}{18} \]

4. (10 points) Solve the differential equation for its general solution.

\[ y'' + 4y = 3 \sin(2x) \]

Let \[ y = A \cos(2x) + B \sin(2x). \]

Since \( \sin(2x) \) duplicates in \( y' \), let \( y_p = x(A \cos(2x) + B \sin(2x)) \)

be a particular solution. Then

\[ y' = x( -2A \sin(2x) + 2B \cos(2x)) + A \cos(2x) + B \sin(2x) \]

\[ y'' = x(-4A \cos(2x) - 4B \sin(2x)) + (-2A \sin(2x) + 2B \cos(2x)) \]

\[ -2A \sin(2x) + 2B \cos(2x) \]

\[ = x(-4A \cos(2x) - 4B \sin(2x)) - 4A \sin(2x) + 4B \cos(2x) \]

Then \( y'' + 4y = 3 \sin(2x) \) gives

\[ x(-4A \cos(2x) - 4B \sin(2x)) - 4A \sin(2x) + 4B \cos(2x) + 4A \cos(2x) \]

\[ + 4B \sin(2x) = 3 \sin(2x). \]

\[ -4A \cos(2x) - 4B \sin(2x) - 4A \sin(2x) + 4B \cos(2x) + 4A \cos(2x) + 4B \sin(2x) = 3 \sin(2x). \]

Therefore \( B = 0 \) and \( A = -\frac{3}{4}. \)
5. (12 points) Solve the differential equation for its general solution by using the method of variation of parameters.

\[ A.E.: \quad m^2 - 2m + 1 = 0 \quad y'' - 2y' + y = \frac{e^x}{1 + x^2} \quad \text{A.E.} \]

\[(m-1)^2 = 0 \]
\[ m = 1,1 \implies y_c = c_1 e^x + c_2 xe^x \]

The independent solutions are \(e^x, xe^x\).

\[ y_p = e^x u_1 + xe^x u_2, \text{where } u_1, u_2 \text{ are given by} \]

\[ u_1 = \begin{vmatrix} x & 1 \\ xe^x & e^x \end{vmatrix} = -\frac{x e^{2x}}{1 + x^2} \quad \text{and} \quad u_2 = \begin{vmatrix} e^x & 1 \\ xe^x & e^x \end{vmatrix} = \frac{xe^{2x}}{1 + x^2} \]

\[ u_1 = -\frac{x}{1 + x^2} \implies u_1 = -\int \frac{x}{1 + x^2} \, dx = -\frac{1}{2} \ln(1 + x^2). \]

\[ u_2 = \frac{1}{1 + x^2} \implies u_2 = \int \frac{1}{1 + x^2} \, dx = \tan^{-1} x. \]

6. (2+10=12 points) Consider the system of differential equations.

\[ \frac{d^2 x}{dt^2} = 4y + e^t \]
\[ \frac{d^2 y}{dt^2} = 4x - e^t \]

(a) What is the total number of arbitrary constants in the general solution?

Four = two + two

(b) Solve the system of the differential equations by elimination method.

\[ D^2 x - 4y = e^t \quad (1) \]
\[ 4y - 4x = -e^t \quad (2) \]

Operating on (1) by \( D \), multiplying (2) by 4 and adding together,

\[ D^4 x - 4D^2 y = e^t \]
\[ 4D^2 y - 16x = -4e^t \]

\[ D^4 x - 16x = -3e^t \]

A.E.: \( m^4 - 16 = 0 \implies (m^2 - 4)(m^2 + 4) = 0 \implies m = \pm 2, \pm 2i \)

\[ x_c = c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos(2t) + c_4 \sin(2t). \]

Let \( y_p = A e^t \).
Then \( x^{(4)} - 16x = -3e^t \) gives
\[
Ae^t - 16Ae^t = -3e^t
\]
\[-15A = -2 \Rightarrow A = \frac{2}{15}.
\]
Therefore
\[
\begin{align*}
x &= c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos(2t) + c_4 \sin(2t) + \frac{1}{5} e^t.
\end{align*}
\]
To find \( y \), we substitute this expression for \( x \) into (1).
\[
+4c_1 e^{-2t} + 4c_2 e^{2t} - 4c_3 \cos(2t) - 4c_4 \sin(2t) + \frac{1}{5} e^t
\]
\[-4y = e^t.
\]
Solving this equation for \( y \),
\[
y = c_1 e^{-2t} + c_2 e^{2t} - c_3 \cos(2t) - c_4 \sin(2t) + \frac{1}{4} \left( \frac{1}{5} e^t - e^t \right)
\]
\[
y = c_1 e^{-2t} + c_2 e^{2t} - c_3 \cos(2t) - c_4 \sin(2t) - \frac{1}{5} e^t
\]
The symbols \( c_1, c_2, c_3, c_4 \) are arbitrary constants.
\[ x(t) = \frac{\sqrt{5}}{2} e^{-2t} \sin(4t + 4.23) \]

\[ x'(t) = -\sqrt{5} e^{-2t} \sin(4t + 4.23) + \frac{\sqrt{5}}{2} e^{-2t} 4 \cos(4t + 4.23) \]

\[ = e^{-2t} \left[ -\sqrt{5} \sin(4t + 4.23) + 2\sqrt{5} \cos(4t + 4.23) \right] \]

When \( 4t + 4.23 = n\pi \),

\[ x'(t) = e^{-2t} 2\sqrt{5} \cos(n\pi) > 0 \quad \text{for} \quad n = 0, 2, 4, \cdots \]

\[ < 0 \quad \text{for} \quad n = 1, 3, 5, \cdots \]

Use \( n = 3 \)

\[ 4t = 3\pi - 4.23 \Rightarrow t = \frac{3\pi - 4.23}{4} \]

\[ = 1.31 \text{ sec.} \]
7. **(6+3+3=12 points)** A mass weighing 64 pounds stretches a spring 0.32 foot. The mass is initially released from a point 8 inches above the equilibrium position with a downward velocity of 5 ft/sec.

(a) Find the differential equation of motion and solve it.

\[ k = \frac{64}{132} = 0.476 \quad \text{mass, } m = \frac{W}{g} = \frac{64}{32} = 2 \]

\[ m x'' + kx = 0 \Rightarrow 2 x'' + 200x = 0 \Rightarrow x'' + 100x = 0 \]

\[ x_c = c_1 \cos(10t) + c_2 \sin(10t) \]

**Initial conditions:**

\[ x(0) = \frac{-8}{12} = -\frac{2}{3} \]

\[ x'(0) = 5 \]

\[ x(0) = -\frac{2}{3} \Rightarrow -\frac{2}{3} = c_1 \]

\[ x'(0) = 5 \Rightarrow 5 = -10 c_1 \sin(0) + 10 c_2 \cos(0) = 10 c_2 \Rightarrow c_2 = \frac{1}{2} \]

**Therefore**

\[ x(t) = -\frac{2}{3} \cos(10t) + \frac{1}{2} \sin(10t) \]

\[ x(t) = \frac{5}{6} \sin(10t - 0.93), \text{ where } -\frac{2}{3} = A \cos \theta, \frac{1}{2} = A \sin \theta, \tan^{-1}\left(-\frac{A}{\frac{1}{2}}\right) \approx -0.93 \]

(b) What are the amplitude and period of motion?

Amplitude = \( \frac{5}{6} \), period = \( \frac{2\pi}{10} = \frac{\pi}{5} \)

(c) At what time does the mass pass through the equilibrium position heading downward for the second time?

Setting \( x = 0 \) gives times for equilibrium position.

\[ 0 = \frac{5}{6} \sin(10t - 0.93) \Rightarrow 10t - 0.93 = n\pi; \quad n = 0, 1, 2, 3, \ldots \]

\[ x'(t) = \frac{5}{6} \cdot 10 \cos(10t - 0.93) = \frac{25}{3} \cos(10t - 0.93) \]

\[ = \frac{25}{3} \cos(n\pi) = \frac{25}{3} (-1)^n \]

The times at the equilibrium position heading downward are:

\[ t = \frac{n\pi + 0.93}{10}, \quad \text{if } n \text{ is even} \]

\[ t = \frac{-25}{3} \quad \text{if } n \text{ is odd} \]

For the second time, \( n = 2 \)

\[ t = \frac{2\pi + 0.93}{10} = 0.72 \text{ seconds} \]
8. \((8+4=12 \text{ points})\) A force of 2 pounds stretches a spring 1 foot. A mass weighing 3.2 pounds is attached to the spring, and the system is then immersed in a medium that offers a damping force that is numerically equal to 0.4 times the instantaneous velocity. Suppose that the mass is initially released from rest from a point 1 foot above the equilibrium position.

(a) Find the differential equation of motion and solve it.

\[
\text{Spring constant, } k = 2. \quad m = \frac{W}{g} = \frac{3.2}{32} = \frac{1}{10} \text{ slugs}
\]

\[
\begin{align*}
\dot{x} & = -\beta x' + kx = 0 \\
\frac{1}{10} \ddot{x} + 0.4x' + 2x & = 0 \Rightarrow \ddot{x} + 4x' + 20x = 0
\end{align*}
\]

\[
A.E.: \quad m\ddot{x} + 4m + 20 = 0 \Rightarrow m = -\frac{4 \pm \sqrt{16 - 80}}{2} = \frac{-4 \pm \sqrt{-64}}{2}
\]

\[
m = \frac{-4 \pm 8i}{2} = -2 \pm 4i
\]

\[
x = e^{2t} (c_1 \cos(4t) + c_2 \sin(4t)). \quad \text{Use} \quad x(0) = -1, \quad x'(0) = 0
\]

\[
-1 = c_2, \quad x'(t) = -2e^{2t} (c_1 \cos(4t) + c_2 \sin(4t))
\]

\[
+ e^{2t} (-4c_1 \sin(4t) + 4c_2 \cos(4t))
\]

\[
0 = -2c_1 + 4c_2 = 2 + 4c_2 \Rightarrow c_2 = -\frac{1}{2}
\]

(b) Find the first time at which the mass passes through the equilibrium position heading upward.

We have from part (a),

\[
x = e^{2t} (-\cos(4t) - \frac{1}{2} \sin(4t))
\]

Put \(-1 = A \sin \theta, \quad -\frac{1}{2} = A \cos \theta \Rightarrow \theta = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}, \quad \tan^{-1}(2) + \pi \text{ (to make it fall in quad III)}
\]

\[
x = \frac{\sqrt{5}}{4} e^{2t} \sin(4t + \tan^{-1}(2) + \pi)
\]
9. \((8+4=12 \text{ points})\) A mass of 1 slug is attached to a spring whose constant is 5lb/ft. Initially, the mass is released 1 foot below the equilibrium position with a downward velocity of 5ft/sec, and the subsequent motion takes place in medium that offers a damping force that is numerically equal to 2 times the instantaneous velocity. Suppose that the mass is driven by an external force equal to \(f(t) = 12\cos(2t) + 3\sin(2t)\).

(a) Find the differential equation of motion and solve it.
\[ x'' + 2x' + 5x = 12\cos(2t) + 3\sin(2t) \]

\[ m = m^2 + 2m + 5 = 0 \Rightarrow m = -1 \pm 2i \]

\[ x_c = e^{-t}(C_1\cos(2t) + C_2\sin(2t)) \]

Let \(y_p = A\cos(2t) + B\sin(2t)\).

We can find that \(A = 0, B = 3\).

The general solution is
\[ x = e^{-t}(C_1\cos(2t) + C_2\sin(2t)) + 3\sin(2t) \]

With \(x(0) = 1, x'(0) = 5\), we get \(C_1 = 0, C_2 = -\frac{1}{2}\).

Therefore \(x = -\frac{1}{2}e^{-t}\sin(2t) + 3\sin(2t)\).

(b) State the transient solution and the steady-state solution. Is the motion of the mass oscillatory as time goes to infinity \((t \to \infty)\)? Explain.

Transient solution = \(-\frac{1}{2} e^{-t} \sin(2t)\)

Steady-state solution = \(3 \sin(2t)\)

Yes, because the steady-state solution is oscillatory.

*Extra-credit (5 points) For the mass/spring system described in problem 7, what is the instantaneous velocity at times when the mass passes through the equilibrium position?

By part (a) of problem 7, \(x(t) = \frac{5}{6} \sin(10t - 0.93)\).

\[ x'(t) = \frac{25}{3} \cos(10t - 0.93) \]

At equilibrium positions, \(\sin(10t - 0.93) = 0 \Rightarrow 10t - 0.93 = n\pi \Rightarrow n = 0, 1, 2, 3, \ldots\)

\[ x'(t) = \frac{25}{3} \cos(n\pi) = \begin{cases} \frac{25}{3} & \text{if } n \text{ is even} \\ -\frac{25}{3} & \text{if } n \text{ is odd} \end{cases} \]