The Lagrangian Equation

INTERMEDIATE MECHANICS

WESLEY QUEEN

Developing a new perspective

- Hero of Alexandria (70 A.D.) - light reflections takes the shortest path
- Pierre de Fermat (1657) - light travels along a path that requires the least time
- Maupertuis (1747) - *action* is minimized through the "wisdom of God"
  - Action is a quantity with dimensions of energy x time
- William Hamilton (1834) –

"Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies."

The Lagrangian

$$L = T - U$$

- T = Kinetic Energy
- U = Potential Energy
- Hamilton's principle asserts that for any given system there is a extremum (maximum or minimum) when the time integral is taken of the difference of Kinetic and Potential Energies.
  - $\int_{t_1}^{t_2}(T - U)dt$ = Min or Max (Minimum in almost all dynamic systems)
- Let S = $\int_{t_1}^{t_2} L\, dt$ "Action"

Calculus of Variations

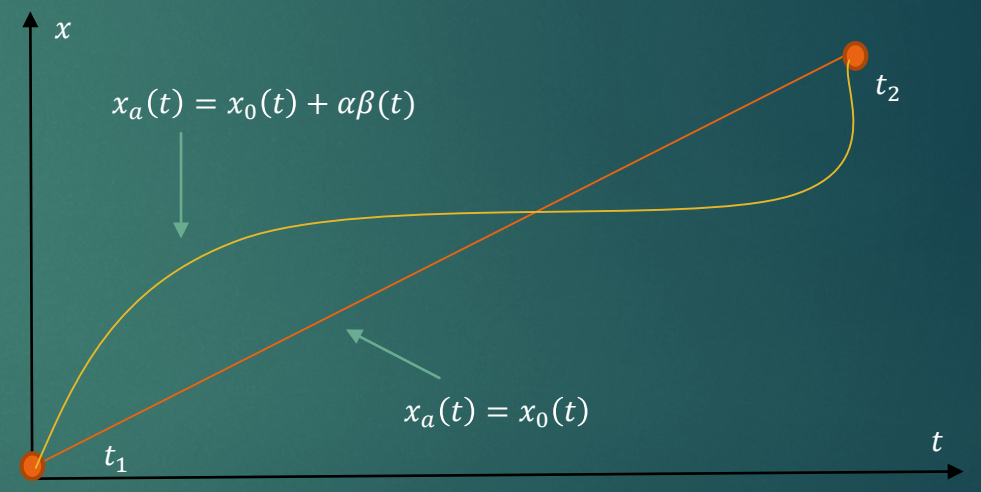
- Begun by Newton
- Developed by Johann Bernoulli, Jakob Bernoulli, and Leonhard Euler
- Important contributions made by Joseph Lagrange, Hamilton, and Jacobi

Leonhard Euler
(1707 - 1783)

Calculus of Variations

Consider a function for any generalized coordinate system $x_a(t) = x_0(t) + \alpha\beta(t)$ where $x_0(t)$ produces a min value for S, α is a number, and $\beta(t)$ is 0 at both end points of our interval.

- *When this is integrated, the t is integrated out and S becomes a number dependent on α, t_1, & t_2.*



How does S depend on α?

$$\frac{\partial}{\partial\alpha} S[x_a(t)] = \frac{\partial}{\partial\alpha}\int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \frac{\partial}{\partial\alpha} L\, dt$$

$$\frac{\partial}{\partial\alpha} S[x_a(t)] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\frac{\partial x_a}{\partial\alpha} + \frac{\partial L}{\partial \dot{x}_a}\frac{\partial \dot{x}_a}{\partial\alpha}\right) dt$$

Use the Chain rule to separate

Calculus of Variations

Use substitution:

From our initial function:
$x_a(t) = x_0(t) + \alpha\beta(t)$

$$\frac{\partial x_a}{\partial \alpha} = \beta \ \& \ \frac{\partial \dot{x}_a}{\partial \alpha} = \dot{\beta}$$

$$\frac{\partial}{\partial \alpha} S[x_a(t)] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\frac{\partial x_a}{\partial \alpha} + \frac{\partial L}{\partial \dot{x}_a}\frac{\partial \dot{x}_a}{\partial \alpha}\right) dt$$

$$\frac{\partial}{\partial \alpha} S[x_a(t)] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\beta + \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\right) dt$$

Calculus of Variations

Use integration by parts:

$$\frac{\partial}{\partial\alpha}S[x_a(t)] = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x_a}\beta + \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\right)dt$$

$$\frac{\partial}{\partial\alpha}S[x_a(t)] = \int_{t_1}^{t_2}\frac{\partial L}{\partial x_a}\beta dt + \frac{\partial L}{\partial \dot{x}_a}\beta\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a}\right)\beta dt$$

$$\frac{\partial}{\partial\alpha}S[x_a(t)] = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a}\right)\beta dt$$

Since $x_0(t)$ produces a stationary value for S, $\frac{\partial}{\partial\alpha}S[x_a(t)] = 0$

Euler's Equation: $$\frac{\partial L}{\partial x_0} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_0}$$

A falling object

Consider a falling object (*no motion in the x or z dimensions*)

$T = \frac{1}{2}m\dot{y}^2$ $\quad U = mgy$ $\quad L = \frac{1}{2}m\dot{y}^2 - mgy$

<u>Apply Euler's Equation</u>

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$$

$$-mg = \frac{d}{dt}(m\dot{y})$$

$$-mg = m\ddot{y}$$

$$\ddot{y} = -\mathrm{g}$$

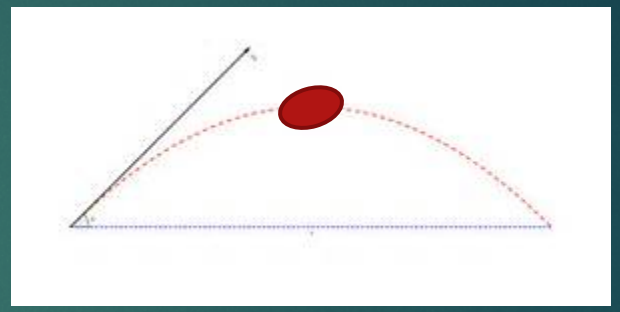
The Lagrangian gives us Newton's second law with respect to gravity.

A projectile

Consider a projectile launched in the positive x direction at some angle θ, where $0 < \theta < 90$.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \qquad U = mgy$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$



<u>X dimension</u>

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

$$0 = \frac{d}{dt}(m\dot{x})$$

$$0 = m\ddot{x}$$

$$\ddot{x} = 0$$

<u>Y dimension</u>

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$$

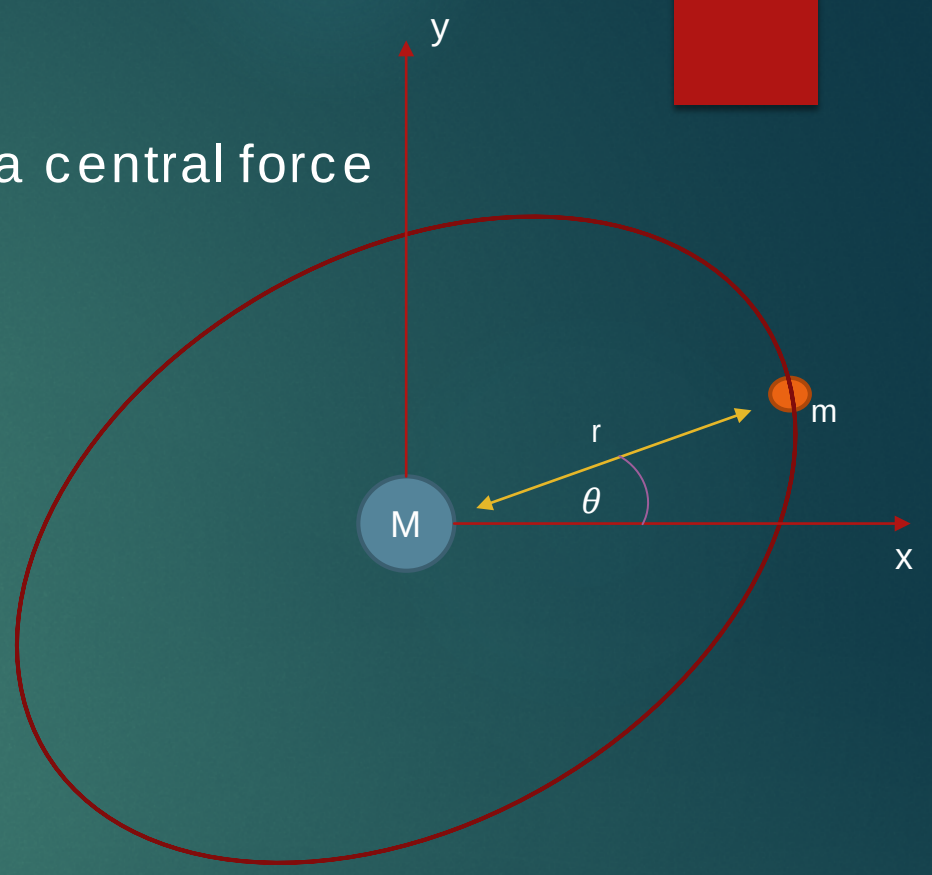
$$-mg = \frac{d}{dt}(m\dot{y})$$

$$-mg = m\ddot{y}$$

$$\ddot{y} = -\mathrm{g}$$

An Orbiting body

Consider a body orbiting about a another body with a central force acting on it.



Position

$x = rcos(\theta)$ $\quad$ $y = rsin(\theta)$

Velocity

$\dot{x} = \dot{r}cos(\theta) - r\dot{\theta}\sin(\theta)$ $\quad$ $\dot{y} = \dot{r}sin(\theta) - r\dot{\theta}cos(\theta)$

The Lagrangian

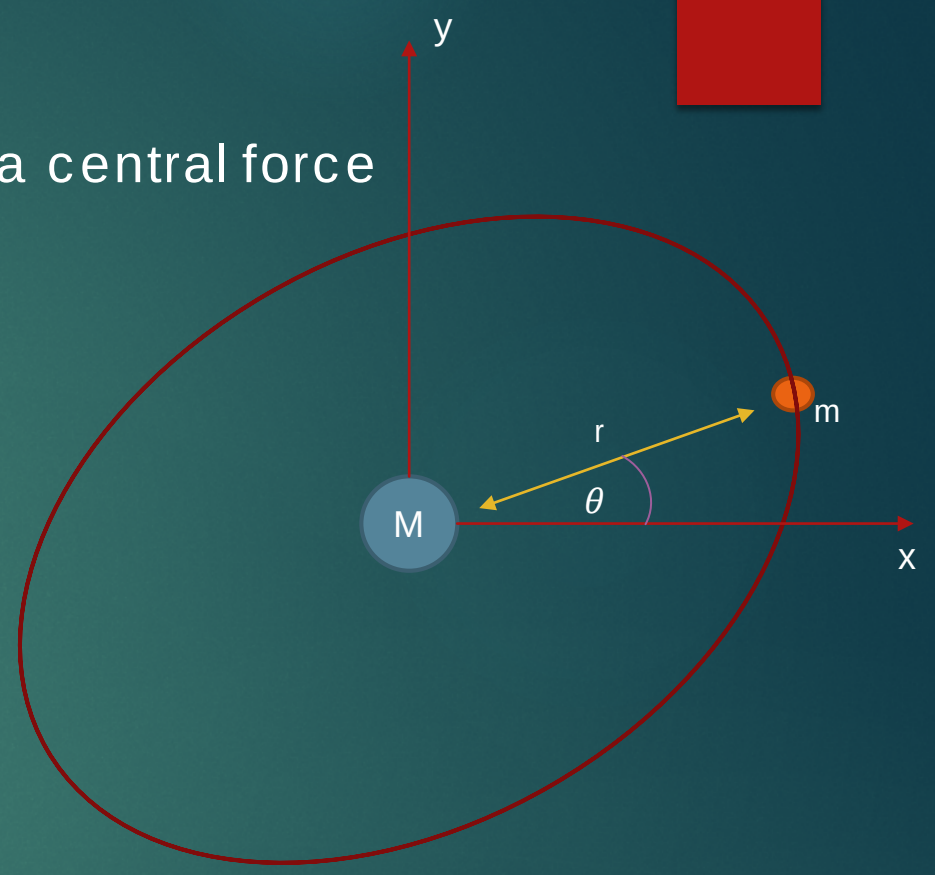
$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ $\quad$ $U = -\frac{GMm}{r}$

$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{GMm}{r}$

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}$

An Orbiting body

Consider a body orbiting about a another body with a central force acting on it.



The Lagrangian

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}$$

Apply Euler's Equations

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$

$$0 = \frac{d}{dt}(\mathrm{m}r^2\dot{\theta})$$

$$\mathrm{m}r^2\dot{\theta} = \mathrm{r\ x\ P}$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}}$$

$$mr\dot{\theta}^2 - \frac{GMm}{r^2} = \frac{d}{dt}(\mathrm{m}\dot{r})$$

$$mr\dot{\theta}^2 - \frac{GMm}{r^2} = m\ddot{r}$$

$$\ddot{r} = -\frac{GM}{r^2} + r\dot{\theta}^2$$

$$a_c = \frac{v^2}{r} = \frac{r^2\dot{\theta}^2}{r}$$

Shows conservation of Angular momentum

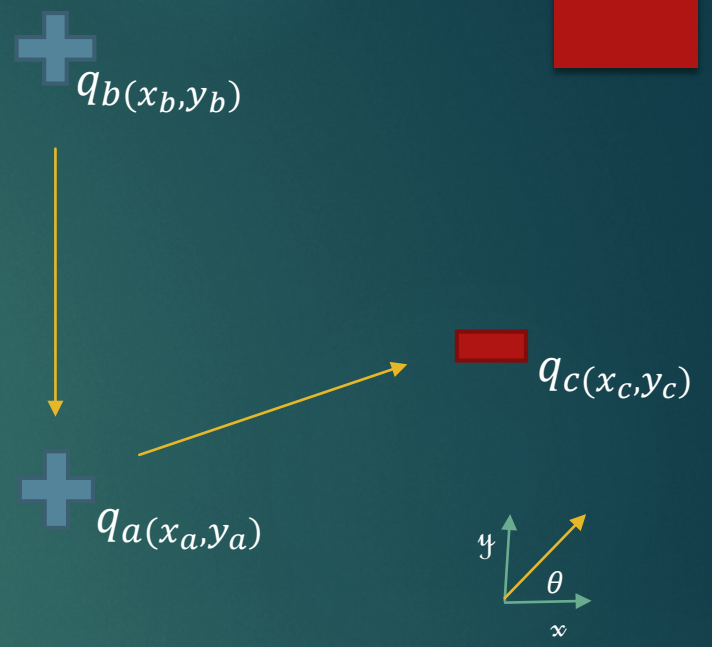
Indicates the acceleration in the direction r is the central force acceleration + the tangential acceleration

Charge interaction

Consider 3 charges interacting in a 2 dimensional plane

$$T = \frac{1}{2}m\dot{x}_a^2 + \frac{1}{2}m\dot{y}_a^2 \qquad U = \frac{kq_aq_b}{\sqrt{(x_b-x_a)^2+(y_b-y_a)^2}} + \frac{kq_aq_c}{\sqrt{(x_c-x_a)^2+(y_c-y_a)^2}}$$

$$L = \frac{1}{2}m\dot{x}_a^2 + \frac{1}{2}m\dot{y}_a^2 - \frac{kq_aq_b}{\sqrt{(x_b-x_a)^2+(y_b-y_a)^2}} - \frac{kq_aq_c}{\sqrt{(x_c-x_a)^2+(y_c-y_a)^2}}$$



X dimension

$$\frac{\partial L}{\partial x_a} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a}$$

$$-\frac{kq_aq_b(x_b-x_a)}{((x_b-x_a)^2+(y_b-y_a)^2)^{3/2}} - \frac{kq_aq_c(x_c-x_a)}{((x_c-x_a)^2+(y_c-y_a)^2)^{3/2}} = m\ddot{x}_a$$

$$-\frac{kq_aq_b}{(x_b-x_a)^2+(y_b-y_a)^2}\cos(\theta) - \frac{kq_aq_c}{(x_c-x_a)^2+(y_c-y_a)^2}\cos(\theta) = m\ddot{x}_a$$

Y dimension

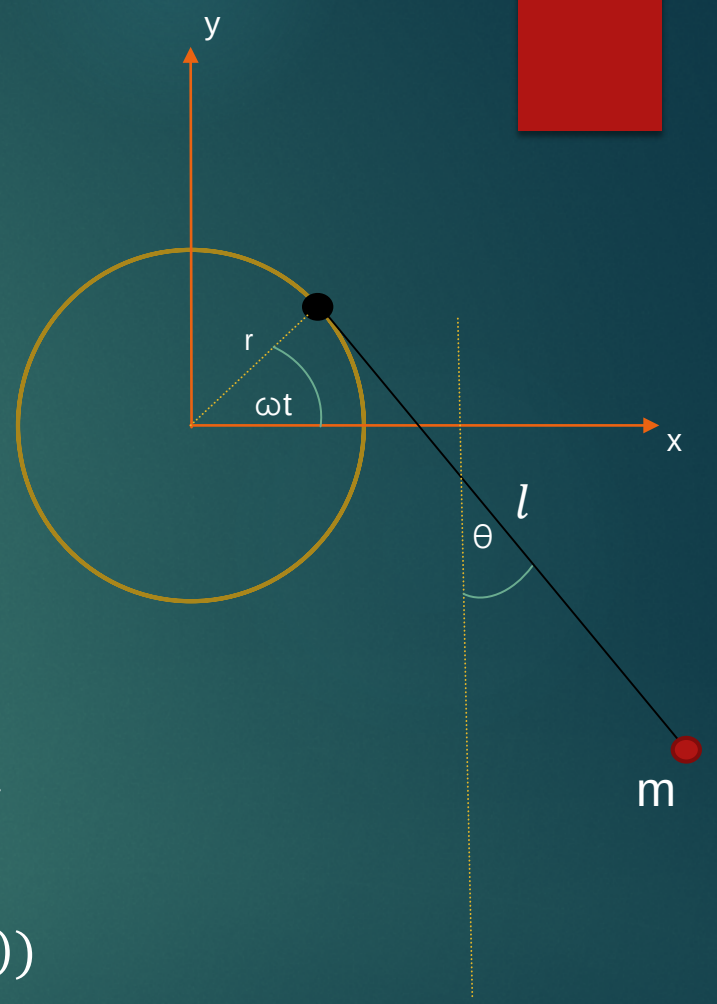
$$\frac{\partial L}{\partial y_a} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_a}$$

$$-\frac{kq_aq_b(y_b-y_a)}{((x_b-x_a)^2+(y_b-y_a)^2)^{3/2}} - \frac{kq_aq_c(y_c-y_a)}{((x_c-x_a)^2+(y_c-y_a)^2)^{3/2}} = m\ddot{y}_a$$

$$-\frac{kq_aq_b}{(x_b-x_a)^2+(y_b-y_a)^2}\sin(\theta) - \frac{kq_aq_c}{(x_c-x_a)^2+(y_c-y_a)^2}\sin(\theta) = m\ddot{y}_a$$

A rotating pendulum

Consider a pendulum of length l moving on a massless ring of radius r and constant angular velocity ω.



Position

$x = rcos(\omega t) + lsin(\theta) \quad y = rsin(\omega t) - lcos(\theta)$

Velocity

$\dot{x} = -r\omega sin(\omega t) + l\dot{\theta}cos(\theta) \quad \dot{y} = r\omega cos(\omega t) + l\dot{\theta}sin(\theta)$

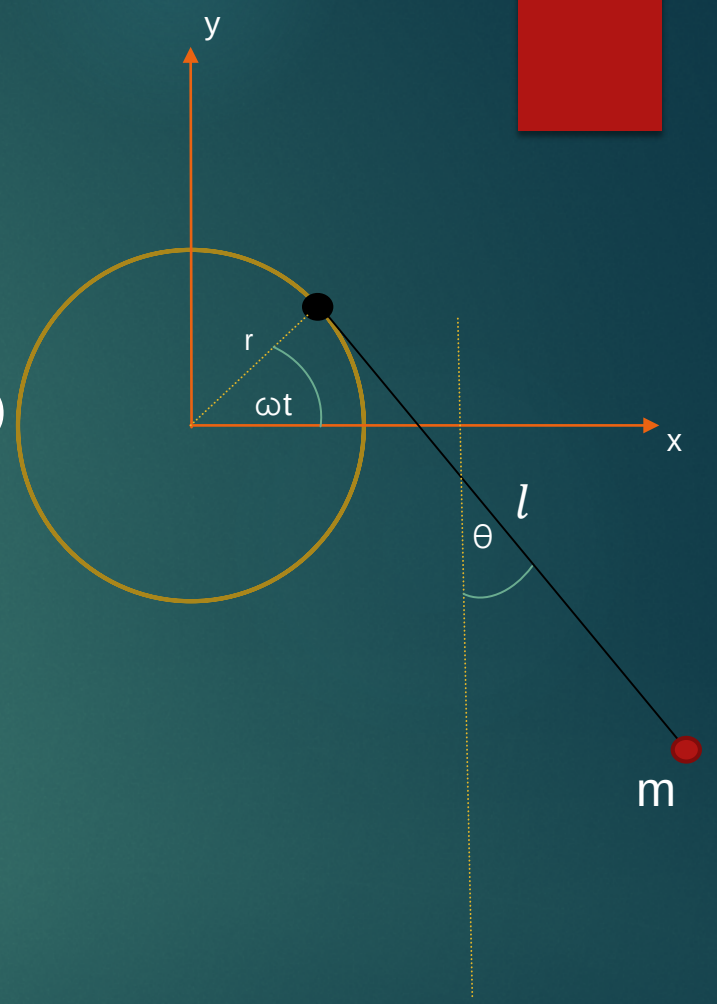
The Lagrangian

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad U = mgy \qquad L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

$L = \frac{1}{2}m[r^2\omega^2 + l^2\dot{\theta}^2 + 2r\omega l\dot{\theta}sin(\theta - \omega t)] - mg(rsin(\omega t) - lcos(\theta))$

A rotating pendulum

Consider a pendulum of length l moving on a massless ring of radius r and constant angular velocity ω.



The Lagrangian

$$L = \frac{1}{2}m[r^2\omega^2 + l^2\dot{\theta}^2 + 2r\omega l\dot{\theta}sin(\theta - \omega t)] - mg(rsin(\omega t) - lcos(\theta))$$

Apply Euler's Equation

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$

$$\frac{\partial L}{\partial \theta} = mr\omega l\dot{\theta}cos(\theta - \omega t) - mglsin(\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}[ml^2\dot{\theta} + mr\omega lsin(\theta - \omega t)]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} + mr\omega l\dot{\theta}cos\,(\theta - \omega t) - mr\omega^2 lcos\,(\theta - \omega t)$$

A rotating pendulum

Consider a pendulum of length l moving on a massless ring of radius r and constant angular velocity ω.

Apply Euler's Equation

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$

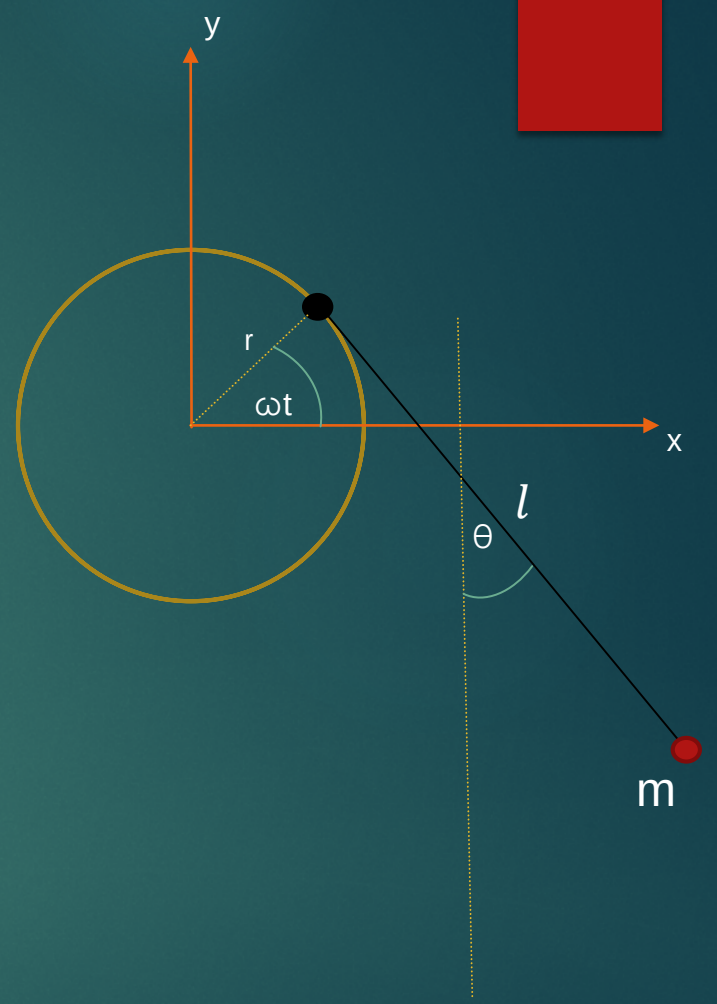
$$\cancel{mr\omega l\dot{\theta}cos(\theta - \omega t)} - mglsin(\theta) = ml^2\ddot{\theta} + \cancel{mr\omega l\dot{\theta}cos\,(\theta - \omega t)} - mr\omega^2 lcos\,(\theta - \omega t)$$

$$ml^2\ddot{\theta} = mr\omega^2 lcos\,(\theta - \omega t) - mglsin(\theta)$$

Solve for $\ddot{\theta}$

$$\ddot{\theta} = \frac{r\omega^2 cos\,(\theta - \omega t)}{l} - \frac{gsin(\theta)}{l}$$

Reduces to simple pendulum when ω = 0.



Lagrangian Benefits

- Deals with derivatives of kinetic and potential energy differences which are invariant to coordinate transformations.
- Can greatly simplify complicated systems.
- Allows us to understand mechanical systems where it is difficult to describe all of the forces explicitly.
- Provides an alternative way to analyze a mechanical system: rather than seeing only cause and effect, we can now analyze a dynamical system by considering action minimization.

Sources

- http://www.people.fas.harvard.edu/~djmorin/chap6.pdf
- https://en.wikipedia.org/wiki/Leonhard_Euler (photo)
- Marion, Thornton. *Classical Dynamics of Particles and Systems, 4th edition,1995. Harcourt Brace & Co.*
- http://physicsinsights.org/rotating_polar_1.html