1 Introduction

Graphs and digraphs are fundamental structures in mathematics and computer science, used to model relationships between objects. This article provides definitions and examples of key concepts related to graphs and digraphs.

2 Definitions

2.1 Graph

A **graph** G is defined as a pair $G = (V, E)$, where:

- V is a set of vertices (or nodes).
- E is a set of edges, which are 2-element subsets of V.

2.2 Digraph

A **digraph** (directed graph) D is defined as a pair $D = (V, A)$, where:

- V is a set of vertices.
- A is a set of arcs (directed edges), which are ordered pairs of vertices.

2.3 Edges and Arcs

An **edge** is a connection between two vertices in a graph, while an **arc** is a directed connection between two vertices in a digraph. For example, in a graph G with vertices $V = \{1, 2, 3, 4\}$, an edge can be represented as $e = \{1, 2\}$. In a digraph D, an arc can be represented as $a = (1, 2)$.

2.4 Loops and Multi-Edges

A **loop** is an edge that connects a vertex to itself. In a digraph, a loop can also be an arc that starts and ends at the same vertex.

A **multi-edge** is a situation where two vertices are connected by more than one edge. In digraphs, this can occur with multiple arcs between the same pair of vertices.

2.5 Simple Graphs

A **simple graph** is a graph that does not contain loops or multiple edges. Each pair of vertices is connected by at most one edge. Unless otherwise specified all graphs in this course are simple.

2.6 Adjacent Vertices

Two vertices u and v are said to be **adjacent** if there is an edge connecting them in a graph or an arc connecting them in a digraph.

The **neighborhood** of a vertex v in a graph G, denoted $N(v)$, is the set of vertices adjacent to v. Consider the graph G with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}$. The neighborhood of vertex 1 is $N(1) = \{2, 3\}$.

In a digraph, the neighborhood can be divided into:

- **In-neighborhood** $N^-(v)$: vertices with directed edges pointing to v.
- **Out-neighborhood** $N^+(v)$: vertices to which v has directed edges.

  Consider the digraph D with vertices $V = \{A, B, C\}$ and arcs $A = \{(A, B), (B, C), (C, A)\}$. The in-degree of vertex B is $d^-(B) = 1$ and the out-degree is $d^+(B) = 1$.

2.7 Degree

The **degree** of a vertex v in a graph is the number of edges incident to v. In a digraph, we distinguish between:

- **In-degree** $d^-(v)$: the number of arcs directed into v.
- **Out-degree** $d^+(v)$: the number of arcs directed out of v.

  For any graph G, $\Delta(G) = \max\{\deg(v)\}$ while $\delta(G) = \min\{\deg(v)\}$.

2.8 Graphic Sequences

A **graphic sequence** is a sequence of non-negative integers that can represent the degrees of the vertices in a simple graph. A sequence $d_1, d_2, \ldots, d_n$ is graphic if there exists a simple graph with degrees equal to these values. The Havel-Hakimi algorithm can be used to determine if a sequence is graphic.

2.9 Havel-Hakimi Theorem

The Havel-Hakimi theorem provides a necessary and sufficient condition for a sequence of non-negative integers to be the degree sequence of a simple graph.

Theorem: A non-increasing sequence of non-negative integers $d_1, d_2, \ldots, d_n$ is graphic if and only if the following process can reduce it to the zero sequence:

1. If all entries are zero, then the sequence is graphic.
2. If the largest entry d_1 is greater than or equal to n (the number of vertices), then the sequence is not graphic.
3. Otherwise, remove d_1 from the sequence and reduce the next d_1 entries by 1. Repeat the process with the new sequence.

2.10 Example

Consider the sequence $d = [4, 3, 3, 1, 0]$.

2.11 Step-by-Step Application

1. **Sort the sequence** (already sorted):
$$d = [4, 3, 3, 1, 0]$$
2. **Remove the first element** $d_1 = 4$ and reduce the next 4 elements by 1:
$$d = [3, 2, 2, 0]$$
3. **Sort the new sequence**:
$$d = [3, 2, 2, 0]$$
4. **Repeat**: Remove $d_1 = 3$ and reduce the next 3 elements by 1:
$$d = [1, 1, 0]$$
5. **Sort**:
$$d = [1, 1, 0]$$
6. **Repeat**: Remove $d_1 = 1$ and reduce the next 1 element by 1:
$$d = [0, 0]$$
7. **Sort**:
$$d = [0, 0]$$

Since we have reduced the sequence to all zeros, the original sequence $d = [4, 3, 3, 1, 0]$ is graphic.

Is the sequence $(3, 3, 2, 2, 1, 1)$ graphic? Explain.

2.12 Conclusion

The Havel-Hakimi theorem is a powerful tool in graph theory for determining whether a degree sequence can correspond to a simple graph. By systematically applying the theorem, one can ascertain the graphic nature of a sequence.

2.13 Handshaking Lemma

The **Handshaking Lemma** states that in any undirected graph, the sum of the degrees of all vertices is equal to twice the number of edges. Mathematically, this can be expressed as:
$$\sum_{v \in V} d(v) = 2|E|$$
where $d(v)$ is the degree of vertex v and $|E|$ is the number of edges in the graph.

Is the sequence $(4, 4, 1, 1, 1, 0)$ graphic? Explain.

2.14 Example of a Graph

Consider the graph G with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}$. The neighborhood of vertex 1 is $N(1) = \{2, 3\}$.

2.15 Example of a Digraph

Consider the digraph D with vertices $V = \{A, B, C\}$ and arcs $A = \{(A, B), (B, C), (C, A)\}$. The in-degree of vertex B is $d^-(B) = 1$ and the out-degree is $d^+(B) = 1$.