1 Binomial Theorem

Expanding the binomial $(x+1)^3$ yields the polynomial x^3+3x^2+3x+1. The coefficients of this polynomial, 1, 3, 3 and 1, happen to also be the values of the binomial coefficients $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}$ and $\binom{3}{3}$. Similarly, expanding $(y+1)^4$ yields $y^4+4y^3+6y^2+4y+1$. Once again, the coefficients of this polynomial happen to be the binomial coefficients $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}$ and $\binom{4}{4}$. This is not a coincidence! The Binomial Theorem presents a formula that allows for quick and easy expansion of $(x+y)^n$ into polynomial form using binomial coefficients.

Theorem 1 *Binomial Theorem: For any real values x and y and non-negative integer n, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.*

The most intuitive proof of the Binomial Theorem is a combinatorial proof.

Now let's focus on using it as a computational tool. The binomial theorem allows for immediately writing down an expansion rather than multiplying and collecting terms. Expanding $(x+3)^4$ into polynomial form yields $(x+3)^4 = \sum_{k=0}^{4} \binom{4}{k} 3^k x^{4-k} = \binom{4}{0} 3^0 x^4 + \binom{4}{1} 3^1 x^{4-1} + \binom{4}{2} 3^2 x^{4-2} + \binom{4}{3} 3^3 x^{4-3} + \binom{4}{4} 3^4 x^{4-4} = 1*1x^4 + 4*3x^3 + 6*9x^2 + 4*27x^1 + 1*81x^0 = x^4 + 12x^3 + 54x^2 + 108x + 81$. Increasing the exponent by one, just adds an additional small computation. When expanding $(3x+2y)^5$ into polynomial form take careful note that both the entire terms $3x$ and $2y$ are raised to respective powers of $5-k$ and k. Thus, $(3x+2y)^5 = \sum_{k=0}^{5} \binom{5}{k} (2y)^k (3x)^{5-k} = \binom{5}{0} (2y)^0 (3x)^{5-0} + \binom{5}{1} (2y)^1 (3x)^{5-1} + \binom{5}{2} (2y)^2 (3x)^{5-2} + \binom{5}{3} (2y)^3 (3x)^{5-3} + \binom{5}{4} (2y)^4 (3x)^{5-4} + \binom{5}{5} (2y)^5 (3x)^{5-5} = 1*1*3^5x^5 + 5*2y^1 3^4 x^4 + 10*2^2 y^2 3^3 x^3 + 10*2^3 y^3 3^2 x^2 + 5*2^4 y^4 3^1 x^1 + 1*2^5 y^5 3^0 x^0 = 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5$. This expansion is generated, quite literally, almost as quick as one can write down the terms.

Since addition is commutative $(x+3)^4 = (3+x)^4$. Don't get hung up on the order of the terms within the binomial. Write the terms in a natural order.

Furthermore, the Binomial Theorem is not restricted to linear powers of variables. The expansion of $\left(5x^2-7\right)^3$ proceeds as smoothly as our previous examples and $\left(5x^2-7\right)^3 = \sum_{k=0}^{3} \binom{3}{k} (-7)^k \left(5x^2\right)^{3-k} = \binom{3}{0} (-7)^0 \left(5x^2\right)^{3-0} + \binom{3}{1} (-7)^1 \left(5x^2\right)^{3-1} + \binom{3}{2} (-7)^2 \left(5x^2\right)^{3-2} + \binom{3}{3} (-7)^3 \left(5x^2\right)^{3-3} = 1*1*125x^6 + 3*(-7)*25x^4 + 3*49*5x^2 + 1*(-7)^3 = 125x^6 - 525x^4 + 735x^2 - 343$.

This process is even faster if only a specific term of the expansion is desired. In order to find the coefficient of x^6 in the expansion of $(2x+5)^9$, generation of the entire polynomial is not needed. For $n=9$, x will clearly have degree 6 when $k=6$. Thus, the coveted term is $\binom{9}{6} (2x)^6 5^3$ and the coefficient of x^6 is $\binom{9}{6} (2x)^6 5^3 = 672\,000x^6$. If the coefficient of x^6 is desired in the expansion of $\left(5x^2-2\right)^7$ then we only need to compute the term for $k=3$. This yields $\binom{7}{3} \left(5x^2\right)^3 (-2)^4 = 70\,000x^6$. Note that the coefficient of x^5 (or any other odd power of x) in the expansion of $\left(5x^2-2\right)^7$ would be 0.

$\left(5x^2-2\right)^7 = 78\,125x^{14} - 218\,750x^{12} + 262\,500x^{10} - 175\,000\,x^8 + 70\,000x^6 - 16\,800x^4 + 2240x^2 - 128$

Not every expansion will be just a binomial expansion. Sometimes, we must perform some of the dirty work by hand. For example, find the coefficient of x^4 in the expansion of $\left(4-x^3\right)^5(2x+3)^7$. While the binomial theorem will not do all the work required, it can be used to expand the two binomials. Also, there is no need to expand either binomial beyond x^4 as those terms will not contribute to the coefficient of x^4. The exist only two ways to achieve an x^4 term in our expansion: a cubic term from $\left(4-x^3\right)^5$ and a linear term from $(2x+3)^7$ or a constant term from $\left(4-x^3\right)^5$ and an x^4 term from $(2x+3)^7$. Thus, the coefficient is $\binom{5}{1}\left(-x^3\right)^1(4)^4\binom{7}{1}(2x)^1 3^6+\binom{5}{5}\left(-x^3\right)^0 4^5\binom{7}{4}(2x)^4 3^3 = 2419\,200x^4$.

The binomial theorem can also be used to transform certain complicated summations into simpler forms. Simplifying $5^{22}\sum_{i=0}^{11}\frac{(-3)^{3i}}{25^i}\binom{11}{i} = \sum_{i=0}^{11}(-3)^{3i}5^{22-2i}\binom{11}{i} = \sum_{i=0}^{11}(-3)^{3i}25^{11-i}\binom{11}{i} = \sum_{i=0}^{11}(-27)^i 25^{11-i}\binom{11}{i} = (25-27)^{11} = -2048$ seems much more reasonable than preserving the original form of the expression.

Exploration of the binomial coefficient logically extended to multinomial coefficients. Not surprising is the usage of multinomial coefficients to quickly expand multinomial expressions into polynomial form. Expanding $(x+y+z)^2$ yields $x^2+2xy+2xz+y^2+2yz+z^2$. The coefficient of the xy term is found by selecting one x term, one y term and no z terms. This can be done in $\binom{2}{1,1,0}=2$ ways. Likewise, the w^2x^3yz term of $(5w+x-3y+z-9)^9$ is found by selecting two w terms, three x terms, one y term, one z term and two constant terms. In this example, not just the variable w is squared but the entire term $5w$. The desired term is $\binom{9}{2,3,1,1,2}(5w)^2x^3(-3y)z(-9)^2 = \frac{9!}{2!^2 3!}(5w)^2x^3(-3y)z(-9)^2 = -91\,854\,000w^2x^3yz$.

While the binomial theorem provides a road map to computing coefficients of particular polynomials using enumerative techniques, the reverse is also true. That is, coefficients of polynomials can be used to count different combinatorial configurations. Consider when siblings Mike, Liz and Julia split a bag of 10 identical candies. Being the oldest, Mike always grabs at least five candies. Liz never eats more than two candies and might not take any at all. Julia always takes at least one candy but never more than four. With these restrictions, how many different distributions of the ten candies are possible? With so few candies a case by case analysis of the solutions is possible. Of course, this will not always be the case. It turns out that this type of problem can be modeled with polynomials in the following manner. Let each person's possible choices correspond to a polynomial. Let the coefficient of x^i be 1 if exactly i candies may be selected and 0 otherwise. Since Liz will select either 0, 1 or 2 candies, the representational polynomial for Liz's possible selections is x^2+x+1[1].

Mike	Liz	Julia
9	0	1
8	0	2
7	0	3
6	0	4
8	1	1
7	1	2
6	1	3
5	1	4
7	2	1
6	2	2
5	2	3

[1] Since $x^0=1$, we write the constant term of the polynomial in its more natural form.

Mike	$x^{10}+x^9+x^8+x^7+x^6+x^5$
Liz	x^2+x+1
Julia	$x^4+x^3+x^2+x$

Computing the product of these three polynomials shows that $\left(x^{10}+x^9+x^8+x^7+x^6+x^5\right)\left(x^2+x+1\right)\left(x^4+x^3+x^2+x\right)=x^{16}+3x^{15}+6x^{14}+9x^{13}+11x^{12}+12x^{11}+11x^{10}+9x^9+6x^8+3x^7+x^6$. Note that the coefficient of the term of the number of candies available is the number of solutions to the original problem. Again, this is hardly a coincidence. How is the coefficient for x^{10} computed in the product of these three polynomials? Three terms x^i, x^j and x^k are selected, one from each polynomial such that $i+j+k=10$. Each such different selection uniquely corresponds to a solution in the original problem. Thus, the coefficient of x^{10} is the number of different possible distributions of ten identical candies. If in opening the bag, two candies are dropped on the floor and immediately eaten by the dog, there would then be only six possible distributions of eight candies to Mike, Liz and Julia since six is the coefficient of x^8.

This technique will easily extend to problems that are not small enough to allow a brute force generation of all solutions. The only drawback is the tedium associated with the multiplication of polynomials. Fortunately there exists wide access to sophisticated calculators and mathematical software that will perform these computations quickly. Furthermore, techniques from a second semester calculus course can also be applied to this type of problem and its generalizations. However, that is not the focus of this text.

A bag of Hershey's Assorted Miniatures contains a supply of four different candies: Hershey's Milk Chocolate, Hershey's Dark Chocolate, Krackel and Mr. Goodbar. With an ample supply of each type of candy available, how many different unordered selections of ten candies can be made? Each candy may be selected from 0 to 10 times and in each case the selection corresponds to the polynomial $f(x)=x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1$. The number of different possible distributions is the coefficient of x^{10} in the expansion of $(f(x))^4$ which is 286. Since there are no restrictions in this problem, it is possible to just make an unordered selection with repetition. Here, $n=4$ and $k=10$ for the same total of $\binom{4+10-1}{10}=286$ different possible distributions.

If, however, an ample amount of each candy was unavailable then making an unordered selection with repetition would not yield a solution. If a bag of Hershey's Assorted Miniatures contains five Milk Chocolates, seven Dark Chocolates, three Krackels and one Mr. Goodbar how many different ways can 10 candies be selected from the bag? Here we use for different polynomials corresponding to possible candy selections.

MC	$x^5+x^4+x^3+x^2+x+1$
DC	$x^7+x^6+x^5+x^4+x^3+x^2+x+1$
K	x^3+x^2+x+1
MG	$x+1$

Now there are 39 ways to select 10 candies.[2] If we require at least one of each candy in our selection we just remove the constant term from our four polynomials and find 15 ways.[3]

Suppose six identical books will be offered to a classroom of eleven students. Eight of the students have the option of keeping a copy or not while the remaining three students each have a sibling and may select zero, one or two copies. In how many ways can all six books be distributed? Our representative polynomial is $f(x) = (x+1)^8 \left(x^2+x+1\right)^3$ and the coefficient of x^6 is 1201.

$f(x) = (x+1)^8 \left(x^2+x+1\right)^3 = x^{14} + 11x^{13} + 58x^{12} + 195x^{11} + 468x^{10} + 849x^9 + 1201x^8 + 1346x^7 + 1201x^6 + 849x^5 + 468x^4 + 195x^3 + 58x^2 + 11x + 1$

This time however, suppose we have an abundant supply of forty identical books that will be offered to a classroom of students. Again, eight of the students have the option of keeping a copy or not while the remaining three students each have a sibling and may select zero, one or two copies. In this case each child could take the maximum number of books. So we could distribute anywhere from zero to fourteen books. How many ways can this be done? Our representative polynomial is still $f(x) = (x+1)^8 \left(x^2+x+1\right)^3$. Now we want the sum of the coefficients in the polynomial expansion of $f(x)$ which is $f(1) = 6912$.

Exercise 2 *Use the Binomial Theorem to expand each of the following into polynomial form.*
i. $(2+x)^5$
ii. $(1-y)^8$
iii. $(3+2x)^4$
iv. $(-2-3z)^5$
v. $(7x+5y)^4$
vi. $\left(1+x^2\right)^5$
vii. $\left(2-4w^3\right)^4$

Exercise 3 *Use the Binomial Theorem to expand each of the following into polynomial form.*
i. $(2+x)^6$
ii. $(z-1)^5$
iii. $(2x-3)^5$
iv. $(2-3w)^4$
v. $(-2w-3z)^5$
vi. $\left(5y-x^2\right)^4$
vii. $\left(2x^4-9x^3\right)^4$

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[2] $\left(x^5+x^4+x^3+x^2+x+1\right)\left(x^7+x^6+x^5+x^4+x^3+x^2+x+1\right)\left(x^3+x^2+x+1\right)(x+1) = x^{16}+4x^{15}+9x^{14}+16x^{13}+24x^{12}+32x^{11}+39x^{10}+44x^9+46x^8+44x^7+39x^6+32x^5+24x^4+16x^3+9x^2+4x+1$

[3] $\left(x^5+x^4+x^3+x^2+x\right)\left(x^7+x^6+x^5+x^4+x^3+x^2+x\right)\left(x^3+x^2+x\right)(x) = x^{16}+3x^{15}+6x^{14}+9x^{13}+12x^{12}+14x^{11}+15x^{10}+14x^9+12x^8+9x^7+6x^6+3x^5+x^4$

Exercise 4 *Use the Binomial Theorem to find the coefficient of x^5 in the polynomial expansion of each of the following.*
i. $(2+x)^8$
ii. $(2+x)^9$
iii. $(2-x)^{10}$
iv. $(4+5x)^7$
v. $(17-13x)^4$
vi. $\left(3x^2-2\right)^7$

Exercise 5 *Use the Binomial Theorem to find the coefficient of x^4 in the polynomial expansion of each of the following.*
i. $(3-x)^7$
ii. $(3-x)^9$
iii. $(2x-5)^{10}$
iv. $(2x+8)^4$
v. $\left(2x^2-1\right)^5$
vi. $\left(3x^3-2\right)^7$

Exercise 6 *Use the Binomial Theorem to find the coefficient of x^4 in the polynomial expansion of each of the following.*
i. $(x+2)^5(2x-1)$
ii. $\left(x^2-1\right)^8(3x+2)^7$

Exercise 7 *Use the Binomial Theorem to find the coefficient of x^7 in the polynomial expansion of each of the following.*
i. $(x-5)^8(x+2)^2$
ii. $(2x-2)^{10}\left(x^3-1\right)^5$

Exercise 8 *Use the binomial theorem to prove* $\sum_{k=0}^{n}\binom{n}{k}=2^n$.

Exercise 9 *Use the binomial theorem to prove* $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$.

Exercise 10 *Compute* $\sum_{k=0}^{n}(-2)^k\binom{n}{k}$ *for odd* n.

Exercise 11 *Simplify* $\sum_{k=0}^{n}c^k\binom{n}{k}$.

Exercise 12 *Find the coefficient of x^3y in the expansion of*
i. $(x+3y+1)^5$
ii. $(2x-y+3)^7$
iii. $(3y+2x-4z+2)^9$

Exercise 13 *Find the coefficient of* x^2y^3 *in the expansion of*
i. $(x+y+3)^5$
ii. $(3x+y-10)^9$
iii. $\left(5x^2-8y+10z-3\right)^{14}$

Exercise 14 *Find the coefficient of* xy^2z^3 *in the expansion of*
i. $(2x-5y+z)^6$
ii. $(-3x-4y+z-5)^8$
iii. $\left(4x+2y^2-z^2+13\right)^{10}$

Exercise 15 *Find the coefficient of* $w^2xy^2z^3$ *in the expansion of*
i. $(x+2y-z+2w)^8$
ii. $(3w+x-5y+z-1)$
iii. $\left(3v-w+2x-y^2+z-5\right)^{15}$

Exercise 16 *The Jones family is going to fill up their cooler to the its twelve can capacity with sodas from their fridge which currently contains twelve packs of Coke and Diet Coke, six-packs of Cherry Coke and Sprite, three cans of Pepsi and two cans of Mountain Dew. If both parents will put at least two cans of Diet Coke each into the cooler, how many different ways can the Jones family cooler be filled with drinks?*

Exercise 17 *Susan has fifteen pens that are identical to one another except for the color of their ink. She has 5 red pens, 4 black pens, 4 blue pens and 2 green pens. How many different ways can she select*
i. two pens;
ii. five pens;
iii. six pens while making sure that she selects at least one pen of each color?

Exercise 18 *Four friends intend to consume a bag of twenty M&M candies. For astrological reasons, Amanda always eats three, seven or nine candies. Jen never eats more than five candies and might not have any at all. Jason only eats an odd number of candies. Finally, Kevin always eats at least five candies. Disregarding the color of the candies, how many different distributions exist if the entire bag is consumed?*

Exercise 19 *Greg has three different kinds of dice. The first die is a standard six sided die with sides numbered 1 through 6. The second die is a five sided die with sides numbered 2, 4, 6, 8, and 10. The third die is a seven sided die with sides numbered 1, 2, 3, 5, 8, 13, and 21. How many different ways can a sum of 17 be rolled when rolling*
i. three standard six sided dice;
ii. four standard six sided dice;
iii. three five sided dice;
iv. two six sided dice and one five sided die;
v. one of each type of die?

Exercise 20 *Donald has 13 pennies, 3 nickels, 2 dimes and a quarter on his desk. How many different ways can he select four coins?*

Exercise 21 *Donald has 13 pennies, 3 nickels, 2 dimes and a quarter on his desk. How many different ways can he select any number of coins totaling 30 cents?*

Exercise 22 *Grimaldi Text Section 1.3: 16, 22, 23, 28, 30*