1 Fibonacci Numbers

$F_0 = 0$
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$F_6 = 8$
$F_7 = 13$
$F_8 = 21$
$F_9 = 34$
$F_{10} = 55$
$F_{11} = 89$
$F_{12} = 144$
$F_{13} = 233$
$F_{14} = 377$
$F_{15} = 610$

Induction is a powerful and easy to apply tool when proving identities about recursively defined constructions. One very common example of such a construction is the Fibonacci sequence. The Fibonacci sequence is recursively defined by

$$\begin{aligned} &F_0 = 0; \\ &F_1 = 1; \\ &F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2. \end{aligned}$$

Theorem 2.2.4 looks at the sum of the Fibonacci numbers with even indices. Returning to the technique used in Theorem 2.2.1, the last term in the summation is separated from the summation in order to use the induction hypothesis.

Theorem 1 *For* $n \in \mathbb{Z}^+$, $\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1$.

Proof. *For* $n = 1, \sum_{k=0}^{1} F_{2k} = F_0 + F_2 = 1 = F_3 - 1$ *. This establishes the base case. Next, assume that the statement is true for* n *(i.e.* $\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1$*) and show that the statement is true for* $n+1$ *(i.e.* $\sum_{k=0}^{n+1} F_{2k} = F_{2n+3} - 1$*). Begin with* $\sum_{k=0}^{n+1} F_{2k}$ *and rewrite the summation with the last term separated out just as in the other induction proofs that involve an indexed sum.*
$\sum_{k=0}^{n+1} F_{2k} = \sum_{k=0}^{n} F_{2k} + F_{2n+2}$
$= F_{2n+1} - 1 + F_{2n+2}$ *by the induction assumption*
$= F_{2n+3} - 1$ *by the definition of the Fibonacci sequence*
■

Another recursively defined construction arises from a counting problem. A group of friends, which includes numerous mathematicians and non-mathematicians, are gathering for a social occasion. At one point, some collection of n of the partygoers will sit in n consecutive chairs. The host does not want two mathematicians sitting next to one another because they will talk shop all night long. Let s_n be the number of ways of seating n people in these n chairs with the aforementioned restriction. When determining the first few values of n and s_n, let □ represent a non-mathematician and ■ represent a mathematician.

n	seatings	s_n
1	□ ■	2
2	□□ ■□ □■	3
3	□□■ ■□■ □□□ □■□ ■□□	5
4	□□□■ □■□■ ■□□■ □□■□ ■□■□ □□□□ □■□□ ■□□□	8

Certainly this sequence appears to be very Fibonacci-like where $s_1 = 2$, $s_2 = 3$ and $s_n = s_{n-1} + s_{n-2}$ for $n \geq 3$. While we are confident about the first two values of s_n, we still must prove the recurrence that $s_n = s_{n-1} + s_{n-2}$ for $n \geq 3$. To prove this relation we focus on the type of person seated last. When seating n individuals, □ can be appended to all seatings of $n-1$ people to cover all the possible ways the arrangement can end with □. While ■ cannot be appended to just any arrangement of $n-1$ people, □■ can be appended to any arrangement of $n-2$ individuals. This takes care of all the ways to end an arrangement with ■. Thus, $s_n = s_{n-1} + s_{n-2}$ for $n \geq 3$.

Exercise 2 *Without using induction show* $F_{n+4} = 3F_{n+1} + 2F_n$ *for* $n \geq 0$.

Exercise 3 *Use induction to prove* $\sum_{k=0}^{n} F_k = F_{n+2} - 1$ *for* $n \geq 0$.

Exercise 4 *Use induction to prove* $\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$ *for* $n \geq 0$.

Exercise 5 *Find and prove the correctness of a formula for* $\sum_{k=0}^{n} F_{2k+1}$ *for* $n \geq 0$.

Exercise 6 *Without using induction prove that the greatest common divisor of any two consecutive Fibonacci numbers is 1.*

Assume there is a pair of consecutive Fionacci numbers divisible by some $d > 1$. What will now be true of the entire sequence?

Exercise 7 *A group of friends, which includes numerous mathematicians and non-mathematicians, are gathering for a social occasion. At one point, some collection of* n *of the partygoers will sit in* n *consecutive chairs. We always want at least two mathematicians sitting next to one another so they will have someone to talk shop to and we always want at least two non-mathematicians sitting next to one another. Furthermore, each row will always begin with a non-mathematician but may end with either type of individual. Let* s_n *be the number of ways of seating* n *people in these* n *chairs. Compute and construct all possible arrangements for all values up to* $n = 5$. *Find and prove the correctness of a recursive formula for* s_n.

Exercise 8 *At a gathering there are* n *chairs and some collection of people (including possibly none) will sit in the seats but there will always be at least one empty chair between any two people. Let* a_n *be the number of antisocial ways to seat some number of people in these* n *seats as described. Compute* a_n *and construct all possible arrangements for all values up to* $n = 4$. *Find and prove the correctness of a recursive formula for* a_n.

Exercise 9 *Let* $A = \{1, 2, ..., n\}$. *Let* S_n *be the collection of subsets of* A *with no consecutive numbers. Prove* $|S_n| = F_{n+2}$.

Exercise 10 *Explain how Exercises 8 and 9 are related.*