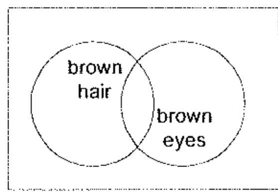
5.1 The Principle of Inclusion/Exclusion

In a group of 45 people, 23 have brown hair and 18 have brown eyes and 6 have both brown hair and brown eyes. How many have either brown hair or brown eyes? The immediate response would be to answer $23 + 18 = 41$ people have either brown hair or brown eyes. This approach counts an individual who has both brown hair and brown eyes twice.



To correct this over-counting it is necessary to subtract the number of individuals who have both brown hair and brown eyes. This will now count everyone who has either brown hair or brown eyes exactly once. There are $23 + 18 - 6 = 35$ people who have either brown hair or brown eyes. How many have neither brown hair nor brown eyes? If 35 people have at least one of brown hair or brown eyes then $45 - 35 = 10$ have neither brown hair nor brown eyes. This suggests the following theorem.

Theorem 5.1.1: For any sets A and B in a collection of n objects, the number of objects in A or B is $|A \cup B| = |A| + |B| - |A \cap B|$.

Corollary 5.1.1: The number of objects that are in neither A nor B is $n - |A \cup B| = n - |A| - |B| + |A \cap B|$.

The next example illustrates Corollary 5.1.1. In one night at a local pizza delivery, 173 pizzas were ordered with pepperoni, 113 pizzas with mushrooms and 67 pizzas were ordered with both pepperoni and mushrooms. If 542 pizzas were ordered then how many pizzas had neither pepperoni nor mushrooms? There were $173 + 113 - 67 = 219$ pizzas with at least one of pepperoni or mushrooms. This implies that there were $542 - 219 = 323$ pizzas with neither pepperoni nor mushrooms.

These concepts can be easily extended to any number of sets.

Theorem 5.1.2 The Principle of Inclusion/Exclusion: For any sets $A_1, A_2, \ldots, A_n$, the number of objects in at least one set is

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_i |A_i| - \sum_{\substack{i,j \\ i \neq j}} |A_i \cap A_j| + \sum_{\substack{i,j,k \\ \text{all different}}} |A_i \cap A_j \cap A_k| - + \ldots - +|A_1 \cap A_2 \cap \ldots \cap A_n|$$

Corollary 5.1.2: The number of objects that are in none of the sets $A_1, A_2, \ldots, A_n$ is $n - |A_1 \cup A_2 \cup \ldots \cup A_n|$.

Consider the following example involving convenience store purchases. Of 457 customers at a convenience store on a Monday, 213 bought gasoline, 127 bought a soft drink and 58 bought a newspaper. There were 78 customers who bought both gasoline and a soft drink, 20 who bought both gasoline and a newspaper and 18 who bought both a soft drink and a newspaper. There were 4 customers who bought gasoline, a soft drink and a newspaper. How many customers did not buy gasoline, a soft drink or a newspaper? There are $213+127+58-(78+20+18)+4=286$ customers who bought at least one of gasoline, soft drink or newspaper. Thus, there are $457-286=171$ customers who did not buy gasoline, a soft drink or a newspaper.

The Principle of Inclusion/Exclusion can also be applied to ordered situations. How many different permutations of the letters $a, b, c, d, e, f, g, h, i$ exist that do not contain any of the following sequences: *cab*, *bed*, *hide*, *fig*? There are 9 letters and hence $9! = 362,880$ different permutations of the given letters. How many permutations contain the word *cab*? This is equivalent to the number of permutations the letters d, e, f, g, h, i and the block *cab*. There are $7! = 5,040$ different permutations of these objects, all of which clearly contain the word *cab*. Continue with the remaining words in a similar fashion. Next, pairs of words must be considered. There are three possibilities with two or more words occurring in the same permutation: the two words share no letters, the two words share exactly one letter at the beginning of one word and at the end of the other or the two words share letters is such a way that they cannot both appear in the same permutation. The words *cab* and *fig* are an example of the first case. There are no letters common to both words. There are three letters, d, e, h and two blocks *cab* and *fig* to permute. This can be done in $5! = 120$ ways. The words *cab* and *bed* are an example of the second case. Both words can appear in the same permutation but only in the form *cabed*. Thus, there are four letters, f, g, h, i and one block, *cabed* to permute in $5! = 120$ ways. Finally, there is the case of hide and bed which cannot occur in the same permutation. The chart below indicates all the possible configurations.

word(s) in permutation	objects to permute	number of permutations
cab	cab, d, e, f, g, h, i	7!
bed	bed, a, c, f, g, h, i	7!
hide	$hide, a, b, c, f, g$	6!
fig	fig, a, b, c, d, e, h	7!
cab, bed	$cabed, f, g, h, i$	5!
cab, hide	$cab, hide, f, g$	4!
cab, fig	cab, fig, d, e, h	5!
bed, hide	n/a	0
bed, fig	bed, fig, a, c, h	5!
hide, fig	n/a	0
cab, bed, hide	n/a	0
cab, bed, fig	$cabed, fig, h$	3!
cab, hide, fig	n/a	0
bed, hide, fig	n/a	0
cab, bed, hide, fig	n/a	0

There are $7!+7!+6!+7!-(5!+4!+5!+0+5!+0)+0+3!+0+0-0=15,462$ different permutations that contain at least one of the words *cab*, *bed*, *hide* and *fig*. There are $9!-15,462=347,418$ permutations that do not contain at least one of the words *cab*, *bed*, *hide* and *fig*.

Homework

1. In an introductory economics class, 35 students created portfolios of stocks and tracked their activity through the semester. There were 18 students who put Coke stock in their portfolio and 14 who included Home Depot stock. If 7 students put both Coke and Home Depot stocks into their portfolios then how many students have neither Coke nor Home depot stock in their portfolio?

2. In a survey of 100 people, 73 drink Coke or Pepsi. Of those surveyed, 33 exclusively drink Coke while 17 exclusively drink Pepsi. How many people drink both Coke & Pepsi? How many people drink neither Coke nor Pepsi?

3. Of 200 movie patrons surveyed, 78 always bought candy or popcorn. Of those people, 45 always bought candy while 51 always bought popcorn. How many people always buy both popcorn and candy?

4. Forty students are enrolled in History 102 and 27 students are enrolled in Biology 112. How many students are enrolled in at least one of the two classes if

   i. History 102 and Biology 112 meet at the same time?

   ii. History 102 and Biology 112 do not meet at the same time and there are 15 students who are enrolled in both classes?

5. A small local business has 85 offices. Of these offices, 70 have a computer, 25 have a fax machine and 33 have a paper shredder. There are 20 offices that have both a computer and a fax machine, 27 offices that have both a computer and paper shredder and 15 offices with both a fax machine and paper shredder. There are 12 offices that have a computer, fax machine and paper shredder. How many offices have none of computer, fax machine or shredder?

6. A builder of 279 homes advertises that each home has at least 1 of the following amenities: four side brick, basement or patio deck. There are 179 houses that are four side brick, 125 homes have a basement and 212 have a patio deck. There are 79 homes that are four side brick and have a basement, 127 homes that are four side brick and have a patio deck and 87 homes that have a basement and a patio deck. Beth wants to purchase a home from this builder that has all three amenities. How many different houses can she choose from?

7. At a local high school, 50 students are on the football team, 19 on the basketball team, and 25 on the baseball team. There are 12 students who play both football and

basketball, 18 who play both football and baseball and 7 who play both basketball and baseball. There are 4 students who play all three sports. How many students play on at least one of football, basketball or baseball?

8. Can the following scenario occur? Explain. There are 95 students who play at least one of football, basket ball and baseball. There are 64 football players, 28 basketball players and 29 baseball players. There are 17 students who play both football and basketball, 13 students who play both football and baseball and 12 students who play both basketball and baseball.

9. How many different permutations of the letters $a, b, c, d, h, i, m, o, t$ exist that do not contain any of the following sequences: *cab*, *math*, *moth*?

10. How many different permutations of the letters $a, c, d, f, h, i, m, n, s, t, u$ exist that do not contain any of the following sequences: *cat*, *fun*, *math*, *this*?

11. A circular spinner whose arrow can be spun clockwise contains the numbers one through six. How many different such spinners can be created such that adjacent labels do not differ by one?