1 Multinomial Coefficients

In studying the binomial coefficient, the selection of a k element subset from an n element set partitioned the set into two subsets, one of size k and one of size $n-k$. A natural extension is to find an ordered partition of a set into three or more subsets each of a specified size.

To illustrate such an ordered partition, consider a bag of Hershey's Assorted Miniatures. It contains four different types of candies: Hershey's Milk Chocolate, Hershey's Dark Chocolate, Krackel and Mr. Goodbar. Ten children will each receive one piece of candy. If there are four Milk Chocolates and two of each of the other flavors, how many different ways can the candy be distributed to the children? This problem is a partition of the set of children into four different subsets. The sizes of the four subsets must correspond to the numbers of each candy available. There are four tasks to perform in this case: select the children to receive each of the four types of candy. There are $\binom{10}{4}$ ways to select the four children to receive a Milk Chocolate candy. There are now six children left without candy. There are $\binom{6}{2}$ ways to select two children to receive a Dark Chocolate candy. At this point, four children are without candy and there are $\binom{4}{2}$ ways to select the two children get a Krackel. Finally, there are $\binom{2}{2}$ ways for the last two candies to be given to the last two children. Therefore, the number of ways to distribute the candies is $\binom{10}{4}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 18\,900$.

Abstractly, the ten objects are partitioned into four different categories in fixed quantities, four in the first category and two each in the other three categories. This is a common type of problem. The multinomial coefficient $\binom{n}{k_1,k_2,\ldots,k_r}$ represents the number of ways to partition n distinct objects into r categories with k_i objects in the ith category. The number of ways to distribute the candies to the children in the previous problem is computed by the multinomial coefficient $\binom{10}{4,2,2,2}$.

When computing $\binom{10}{4,2,2,2}$ an interesting pattern of cancellations arises. The product $\binom{10}{4}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{10!}{4!6!} * \frac{6!}{2!4!} * \frac{4!}{2!2!} * \frac{2!}{2!2!}$ has a number of factorials in the numerator and denominator that will cancel with each other. Cancellation of these terms and discarding yields $\frac{10!}{4!2!2!2!}$. This pattern is not a coincidence.

The multinomial coefficient which reduces to and proves the result of Theorem 1.6.1.

Theorem 1 *The multinomial coefficient* $\binom{n}{k_1,k_2,\ldots,k_r} = \frac{n!}{k_1!k_2!\cdots k_r!}$.

To use a multinomial coefficient one must identify the set and the ordered partition of interest. For example, a teacher decides that the grade distribution in a class of ten students should be such that there are an equal number of each of the five letter grades. How many different ways can students be assigned grades to satisfy such a distribution? The set of ten students is to be partitioned into five subsets corresponding to the five letter grades with two students in each subset. Thus, there are $\binom{10}{2,2,2,2,2} = \frac{10!}{2!^5} = 113\,400$ different possible grade distributions.

The partition of the set in the above example is clear, but in other cases it may be more difficult to identify. Recall a previous problem type where we rearranged the letters of a word such as *level*. Initially the strategy considers this a permutation of five distinct letters, resulting in $5! = 120$ arrangements. Of course, the two l's and two e's are indistinguishable and the order they appear in is unimportant in the an arrangement, i.e., the word *eellv* is the same regardless of whether the two e's are switched. Subsequently, the order is divided out for both the two l's and the two e's resulting in $\frac{5!}{2!2!} = 30$ arrangements. Insert a 1! for the v and the form of the multinomial coefficient $\frac{5!}{2!2!1!} = 30$ appears in this solution. But where is the partition? Focusing on the five positions rather than the letters reveals the partition. We place two character positions in the category l, two character positions in the category e and one character position in the category v. With this technique it is easy to count the number of different ways the letters in the word *bookkeeper* can be arranged. This is given by the multinomial coefficient $\binom{10}{1,2,2,3,1,1} = \frac{10!}{2!^2 3!} = 151\,200$.

Problem 2 *Professor McKenzie will administer oral final exams to ten students. Her schedule will allow for one final on Monday, three on Friday and two each on the other three days. How many different ways can the students be scheduled for exams?*

Problem 3 *Twelve different food critics will independently review three new restaurants. Six will review a new Chinese restaurant, four will review a new steak house and, due to financial constraints, only two will review a new five star French restaurant. How many different ways can the critics be assigned restaurants to review?*

Problem 4 *A twelve person club needs to form three committees for membership, publicity and fund-raising. Each committee must have a chairperson. The membership committee will have five members in addition to the chair. The publicity and fund-raising committees will both have two members in addition to their respective chairs.*
i. How many ways can the committees be formed if each club member will serve on exactly one committee?
ii. How many ways can the committees be formed if each club member will serve on exactly one committee but Bud, Mac and Sarah must serve on different committees?
iii. How many ways can the committees be formed if a club member may serve on any number of committees, including none?

Problem 5 *Dr. Jones teaches a class of fifteen archeology students.*
i. How many different ways can students earn the five different possible letter grades?
ii. If the grade distribution will have an equal number of B's and C's, an equal number of A's and D's, the number of F's is half the number of A's and the

number of B's is two and a half times the number of A's, how many different ways can the grades be assigned?[1]

Problem 6 *How many different arrangements exist of the letters in the element*
i. gold;
ii. mercury;
iii. copper;
iv. manganese;
v. phosphorus?

Problem 7 *How many different arrangements exist of the letters in the cat breed*
i. manx
ii. siberian
iii. abyssinian
iv. himalayan
v. siamese

Problem 8 *Find the number of different arrangements of the letters in the following state's names. Be sure to treat uppercase and lowercase letters as distinguishable objects.*
i. Maine;
ii. Nevada;
iii. California;
iv. Connecticut;
v. Tennessee.

Problem 9 *Find the number of different arrangements of the letters in the following state capitol's names. Be sure to treat uppercase and lowercase letters as distinguishable objects.*
i. Boston;
ii. Trenton;
iii. Cheyenne;
iv. Indianapolis;
v. Tallahassee.

Problem 10 *Find the number of different arrangements of the letters and blank space in the cat breed Scottish Fold. Be sure to treat uppercase and lowercase letters as distinguishable objects.*

Problem 11 *Find the number of different arrangements of the letters and blank space in the cat breed Maine Coon.*

Problem 12 *Ten colleagues plan a lunch outing and will take three cars. Two cars seat four people each but the third only seats two. Given that the car owners will drive their own car, how many different ways can people decide in which car to ride?*

[1] Please excuse the completely gratuitous inclusion of algebra.

Problem 13 *Repeat Problem 12 for three cars that seat four people each.*