1 Multiplication Rule

A basic counting technique in combinatorics is the multiplication rule. Consider a club that consists of three individuals: Mark, Kate and Pat. Every club needs a president and a vice-president, so the question becomes how many different elected choices of president and vice-president can be chosen for this small club? In this case, it is very easy to write down all possible selections as demonstrated to the left.

Pres.	V-P
Mark	Kate
Mark	Pat
Kate	Mark
Kate	Pat
Pat	Mark
Pat	Kate

Clearly, there are six different ways to choose a president and vice-president. But, what if the club had twenty members? This brute force method would not be a reasonable approach. It would be difficult to be certain that all election possibilities had been found. There is a need for a more sophisticated counting technique rather than just primitive brute force.

Upon closer analysis, it is clear that there are three people who may be selected as president. For each choice of president, there are then two choices for vice-president. It is no coincidence that $3 * 2 = 6$. If the club consisted of 20 people, there would be $20 * 19 = 380$ ways of choosing the president and vice-president. This is the fundamental idea behind the most important theorem in combinatorics.

Theorem 1 ***Multiplication*** *(or* ***Product****) Rule: If task one can be performed in n_1 ways and task two can be performed in n_2 ways regardless of how task one was performed, then the total number of ways of performing task one and then task two is $n_1 n_2$.*

Example 2 *As an example, suppose a state only needs 12 million unique license plates. Will enough different license plates exist with three uppercase letters followed by three digits? There are six different tasks. The first three tasks are to pick the first, second and third letters. When deciding on the number of ways that the second letter can be selected, the question of repetition arises. Can the second letter be identical to the first? In this case, it is common for license plates to contain multiple copies of the same letter or digit. Hence, repetition is allowed. The last three tasks are to pick the first, second and third digits. The multiplication rule indicates that $26^3 10^3 = 17\,576\,000$ different license plates exist which is sufficient for the state's needs.*

Example 3 *Suppose this state also wishes to not repeat characters in its license plates. Will enough different license plates exist using the scheme of the previous problem but disallowing repetition? Now, the second letter selected must be different from the first letter selected and regardless of the specific letter there are only 25 letters available. The tasks are remain the same but now result in $26 * 25 * 24 * 10 * 9 * 8 = 11\,232\,000$ different license plates. Only a small change in design was made but but the number of plates was reduced over 33% and the supply of license plates is now insufficient.*

Remark 4 *The careful reader of the multiplication rule is now asking what the phrase, regardless of how the first task is performed, means. It means the multiplication rule holds only if regardless of how the first task is performed, there* *are always the same number of choices for performing the second task. This concept is illustrated in the above example. In some problems the counting technique may have to be changed to achieve this. The following example illustrates how the multiplication rule may be applied in different situations by choosing appropriate tasks.*

Example 5 *Consider the twenty person club made up of eight men and twelve women. Suppose a president and vice-president of opposite gender must be selected. There are still twenty choices for president but a sticky point arises when we go to choose a vice-president. On one hand, if a male president is selected then there are twelve choices for a female vice-president. On the other hand, if a female president is chosen then there are eight choices for a male vice-president. In this case, the number of ways task two can be performed changes based on the choice for task one. In order to use the multiplication rule here, the problem must be broken into three tasks rather than two. Task one is to pick a male to serve in office. Task two is to pick a female to serve in office. Finally, task three selects the gender of the president (which automatically selects the other gender to serve as vice-president). Hence, the number of different ways to select a president and vice-president of opposite gender is* $8 * 12 * 2 = 192$. *While the multiplication rule did not work as easily as we might have preferred, ultimately the rule was sufficient for the problem at hand.*

Problem 6 *How many different passwords exist using four lowercase letters followed by two digits (0-9)?*

Problem 7 *How many different passwords exist using four distinct lowercase letters followed by two distinct digits (0-9)?*

Problem 8 *How many different passwords exist using four distinct lowercase or uppercase letters followed by two distinct digits (0-9)?*

Problem 9 *How many different passwords of seven characters exist where each character may be a lowercase letter of the alphabet or a digit?*

Problem 10 *How many different passwords of seven characters exist where each character may be a lowercase or uppercase letter of the alphabet or a digit?*

Problem 11 *How many different ways can three six-sided dice be rolled?*

Problem 12 *How many six digit numbers have no two consecutive digits the same? Think about the difference between a 6 digit number and a 6 digit password.*

2 Exercises

1. A collection of seven distinct coins will be arranged from left to right. There are four heads face up and three tails face up.
   i. How many different ways can the coins be arranged from left to right?
   ii. How many different ways can the coins be arranged from left to right if there can be no consecutive heads?
   iii. How many different ways can the coins be arranged from left to right if all heads must be consecutive and all tails must be consecutive?
   iv. How many different ways can the coins be arranged from left to right if all heads must be consecutive?

2. Two pairs of siblings (two brothers and a brother/sister), and three only children (two men and one woman) sit in a row of seven consecutive seats. How many ways can they be seated:
   i. with no restrictions;
   ii. alternating genders;
   iii. such that the women are all consecutive;
   iv. such that siblings sit next to one another?

3. How many four digit numbers begin and end with the same digit?

3 Multiplication Rule: Subtraction and Factorials

We now introduce some basic terminology. Suppose you wanted to count how many different batting orders exist for the nine positions for an American League baseball team where any position can bat at any spot in the order. This is a direct application of the multiplication rule. There are nine tasks to perform. Pick a position to bat first and then a position to bat second, etc. Clearly there are $9*8*7*6*5*4*3*2*1 = 362\,880$ ways to do this. Each of the 362,880 different orders of these nine positions is called a permutation. The product of all the integers from 1 to k is denoted as k-factorial. The total number of permutations of k objects is k-factorial. Rather than write the product of k integers every time a factorial is needed, an exclamation mark is used to denote the factorial operation.

Definition 13 *The* ***factorial*** *function* $k! = k(k-1)(k-2)\cdots 3*2*1$ *for every positive integer* k. *Furthermore, as a matter of convenience, we define* $0! = 1$. *Note that* $k!$ *grows very quickly as the table illustrates.*

$0! = 1$
$1! = 1$
$2! = 2$
$3! = 6$
$4! = 24$
$5! = 120$
$6! = 720$
$7! = 5040$
$8! = 40\,320$
$9! = 362\,880$
$10! = 3628\,800$

Example 14 *How many different permutations of the characters* $\{a, b, c, d\}$ *exist. There are four characters so there will be different* $4! = 24$ *permutations. All possible permutations are listed to the right.*

abcd	abdc	acdb
acbd	adbc	adcb
bacd	badc	bcad
bcda	bdac	bdca
cabd	cadb	cbad
cbda	cdab	cdba
dabc	dacb	dbac
dbca	dcab	dcba

At times we may not need to order all elements in a set. It is still easy to factorial notation to represent such a count.

Example 15 *How many different permutations of two characters from* $\{a, b, c, d, e\}$ *exist. We can pick these two ordered elements in* $5 * 4 = \frac{5!}{(5-2)!} = \frac{5!}{3!}$ *ways.*

Definition 16 *The number of permutations of* k *objects from a set of* n *objects is* ${}_nP_k = \frac{n!}{(n-k)!}$.

Definition 17 *Veteran wisdom in mathematics states that a technique is something you use over and over again while a trick is something you use once. Students claim any technique that differs slightly from the example is a trick. I'm certain the truth lies somewhere in the middle. Initial problems utilizing the multiplication rule can seem trivial. There are some very standard ways to use the multiplication rule that are not obvious at first glance.*

Another approach to the president and vice-president of opposite gender problem would be to count all the possible ways to select a president and vice-president without regard to gender and then subtract off all the undesirable selections. There are $20*19 = 380$ ways to select a president and vice-president. The unwanted selections are those selections from the same gender. There are $8 * 7 = 56$ ways to select a male president and male vice-president. There are $12*11 = 132$ ways to select a female president and female vice-president. Thus, there are $380 - 56 - 132 = 192$ different ways to select a president and vice-president of opposite gender. Counting all possible cases and then subtracting of forbidden cases is a very useful technique. This technique works exceptionally well when encountering problems involving the form "at least" or "at most."

In our twenty person club, we have eight seniors, four juniors, three sophomores and five freshmen. How man ways can we select a President, Vice-President and Treasurer with at least one senior serving in office? We have $20 * 19 * 18 = 6840$ ways to select these officers. However, some of these selections do not have the required senior among them. How many? There are $12*11*10 = 1320$ selections without a senior. Hence, there are $6840 - 1320 = 5520$ with at least one senior.

4 Exercises

1. Compute each of the following.
   i. $12!$
   ii. $4!5!$
   iii. $5\,(4!)$
   iv. $(2+3)!$
   v. $\frac{5}{4!}$
   vi. $\frac{100!}{98!}$
   vii. $\frac{100!}{98!2!}$
   viii. $\frac{n!}{(n-2)!}$
   ix. $\frac{(n+1)!}{(n-1)!}$

2. Dr. Ian Malcolm is arranging his favorite books about dinosaurs (*Jurassic Park*, *God Creates Dinosaurs, The Lost World of the Dinosaurs* and *Dinosaur Detective*) on his bookshelf. In how many ways can Ian arrange his books from left to right?

3. In an attempt to raise productivity the CANE corporation is scheduled to publicly flog its six least productive employees. In how many different orders can these six employees be made an example of?

4. How many different ways can the letters in the word **riot** be arranged? Find them all.

5. How many different ways can the letters in the word **stygian** be arranged? How many different ways can the letters in the word **stygian** be arranged such that the word **sing** appears in the arrangement?

6. How many different passwords of seven characters exist where each character may be a lowercase letter of the alphabet and contains at least one vowel?

7. True or False? $(n+k)! = n! + k!$ for all positive integers n and k. If true, prove the statement. If false, provide a counterexample to demonstrate that the statement is false.

8. True or False? $(n+k)! = n! * k!$ for all positive integers n and k. If true, prove the statement. If false, provide a counterexample to demonstrate that the statement is false.

9. True or False? $(nk)! = n! * k!$ for all positive integers n and k. If true, prove the statement. If false, provide a counterexample to demonstrate that the statement is false.

10. Prove $\frac{n!}{(n-2)!2!}$ is an integer for all integers $n \geq 2$.