1 Unordered Selections without Repetition

All the examples presented so far have an inherent order implied in the statement of the problem. Virtually every application of the multiplication rule introduces order into a problem, usually in a subtle way. Consider the following two examples where order is not important in the problem.

Example 1 *Two friends are planning to order pizza for dinner. Toppings offered by the restaurant are mushrooms, sausage, pepperoni, anchovies, onions and peppers. How many different combinations of toppings can be ordered for the pizza? There are six potential toppings and for each the only decision is to include the topping on the pizza or not to include the topping on the pizza. This breaks down to six tasks to perform where each task can be performed in two ways. There exist* $2^6 = 64$ *different ways to choose toppings for this pizza. This runs the gamut of a plain pizza (no toppings) to "all the way."*

One of the first steps in solving counting problems is to determine whether or not order is important. And while almost all problems where order is important involve the multiplication rule, there are solutions to problems where order is not important that also employ this rule. Whenever the multiplication rule is used it is crucial to understand the role order is playing in the solution.

Example 2 *Consider the pizza problem again. There are still six toppings-mushrooms, sausage, pepperoni, anchovies, onions and peppers- from which to choose. Now the task is to determine how many pizzas exist with exactly three toppings. The naive approach uses three tasks, pick topping number one, pick a different topping number two and pick a different topping number three. This would be done in ways* $6 * 5 * 4 = 120$. *However, closer analysis is needed.*

topping one	topping two	topping three
onions	peppers	sausage
onions	sausage	peppers
peppers	onions	sausage
peppers	sausage	onions
sausage	onions	peppers
sausage	peppers	onions

These six cases are considered different in the 120 selections because the design of the three tasks implies that peppers as topping one is somehow different from peppers as topping two or as topping three. The multiplication rule has introduced an order to the toppings when no order is required. As any experienced pizza connoisseur will tell you, the order that the toppings go onto the pizza is irrelevant! All that matters is whether the topping is present or absent. One solution to fix this problem is to eliminate the repetition. Each set of three toppings will give rise to $3! = 6$ permutations as the example above does. Thus, the true number of three topping pizzas is $\frac{6*5*4}{3!} = 20$. This example demonstrates the need for a method for picking objects as a set without forcing an ordering of the elements.

The general question we ask is, how many ways can k objects be selected from n objects as a subset with no implied order? Using the above strategy, first k objects are selected. This can be done in $n\,(n-1)\cdots(n-k+1)$ ways. Second, the repetition must be eliminated. Each set of k objects occurs in $k!$ different orders. The total number of ways of selecting a subset of k objects from n objects is $\frac{n(n-1)\cdots(n-k+1)}{k!}$. With a little manipulation this can be written as $\frac{n!}{k!(n-k)!}$ and the result of Theorem 3 follows.

Theorem 3 *The number of ways of choosing k distinct objects as a subset from n distinct objects is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for integers $0 \le k \le n$. The symbol is called the binomial coefficient, and is read "n choose k".*

The binomial coefficient $\binom{n}{k}$ (also commonly written as $C_{n,k}$ or ${}_nC_k$) is the number of combinations of n items taken k at a time. With this notation, the solution to the above pizza problem is $\binom{6}{3} = 20$.

The values of five particular pairs of parameters are simple to determine based solely on the fact that the binomial coefficient counts the number of ways to select k items from n items as a subset. We will establish these values without resorting to the formula of Theorem 3. Rather we focus on the logic of the selections. For all positive integers n, $\binom{n}{0} = 1$ because there is only one way to select the empty set. Likewise, $\binom{n}{n} = 1$ as there is only one way to select the set itself. The binomial coefficient $\binom{n}{1} = n$ since there are n different options for a single selection. Likewise $\binom{n}{n-1} = n$ as there are n different choices for the item to not be selected. Finally, $\binom{n}{k} = 0$ for $k > n$ since one cannot select more items than are available without repetition.

Example 4 *Returning to the twenty person club from the previous section, how many ways can a three person committee be selected if all three individuals are of equal rank and power? This is enumerated by the binomial coefficient twenty choose three since the order in which the three are selected is not important. This results in $\binom{20}{3} = 1140$ possible committees.*

Example 5 *How many subsets of size two from $S = \{a, b, c, d, e, f, g, h\}$ exist? Just as in the pizza problem an unordered selection of two elements is required. This selection can be made in $\binom{8}{2} = 28$ ways.*

Example 6 *Even though the binomial coefficient easily counts unordered combinations, sometimes a different perspective may aid in a solution. Bethany and Tyler will each buy five different pieces of candy from twelve different types of candy. Bethany selects her five pieces by selecting her five favorite types. Bethany does this in $\binom{12}{5} = 792$ different ways. Tyler is somewhat negative and reasons that he will select his five pieces of candy by removing his seven least favorite kinds of candy. The five remaining candies will then be his selection. Tyler can do this in $\binom{12}{7} = 792$ different ways. It is no coincidence that $\binom{12}{5} = \binom{12}{7}$. Designating k objects to be selected from n is actually the same as designating $n-k$ objects from n to not be selected. It will always be the case that $\binom{n}{k} = \binom{n}{n-k}$, the proof of which is left for the reader as Problem* **??**.

Example 7 *How many different arrangements of the letters in the word moon exist? The answer is not* $4!$ *since the letters are not all distinct. One solution is to divide out the repetition caused by the duplicate letters. This approach yields* $\frac{4!}{2!} = 12$ *arrangements. Alternatively, we could first select two locations for the ohs in an unordered fashion and then permute the remaining distinct letters for a total of* $\binom{4}{2} * 2! = 12$ *arrangements.*

mnoo
mono
omno
moon
omon
oomn
nmoo
nomo
onmo
noom
onom
oonm

Example 8 *How many three digit numbers exist such that each digit is larger than the digit to its right? This problem seems to be tailor-made for the multiplication rule where order is important. Looks can be deceiving. First, note that the digits must be distinct since no digit is larger than itself. Second, given any three different digits, there exists exactly one way to arrange them so that each digit is larger than the digit to its right. If the digit set* $\{3, 8, 9\}$ *is selected then only the number 983 can be constructed to satisfy the given restrictions. The original problem reduces to an unordered selection of three distinct digits from 0 through 9. This can be done in* $\binom{10}{3} = 120$ *ways. What first appeared to be a problem where order is important turns out not to be.*

Theorem 9 *For integers* n *and* k, $\binom{n}{k} = \binom{n}{n-k}$.

Proof. The proof begins by applying the formula for binomial coefficients to the right-hand side of the equation and simplifying. ■

$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$

Theorem 10 *For integers* n *and* k, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof. ■

2 Exercises

1. In a forty person club, how many different ways can a
   i. president, vice-president, and secretary be selected;
   ii. three person committee be chosen?

2. In our twenty person club, we have eight seniors, four juniors, three sophomores and five freshmen. How man ways can we select a five person committee with at least one senior?

3. A restaurant offers pizza with eight different toppings available. If a large, medium or small pizza with any combination of toppings (but no toppings may be repeated, i.e. no double pepperoni) and any number of toppings can be selected then how many different pizzas with exactly four toppings can be ordered?

4. How many different ways can the letters in the following words be arranged?
   i. house
   ii. building
   iii. construction

iv. structures

5. In the game of Clue, there are six suspects (Col. Mustard, Prof. Plum, Mr. Green, Mrs. Peacock, Miss Scarlet and Mrs. White), six possible weapons (Knife, Candlestick, Revolver, Rope, Lead Pipe and Wrench) and nine locations (Hall, Lounge, Dining Room, Kitchen, Ball Room, Conservatory, Billiard Room, Library and Study). The murder of Mr. Boddy was committed by one suspect, with one weapon in one location.
   i. How many different possible ways could the murder have been committed?
   ii. How many ways could Miss Scarlet have committed the murder?

6. Jason's CD collection consists of five different rock CD's, three different jazz CD's, two different blues CD's, two different classical CD's and a single folk CD. Jason is planning a trip and randomly selects four CD's.
   i. How many different ways can this be done?
   ii. How many different ways can this be done if the folk CD must be one of the four?
   iii. How many different ways can this be done if Jason will take exactly one rock CD?
   iv. How many different ways can this be done if Jason will take at least one rock CD?

7. A collection of seven indistinguishable coins will be arranged from left to right. There are four heads face up and three tails face up.
   i. How many different ways can the coins be arranged from left to right?
   ii. How many different ways can the coins be arranged from left to right if there can be no consecutive heads?
   iii. How many different ways can the coins be arranged from left to right if all heads must be consecutive and all tails must be consecutive?
   iv. How many different ways can the coins be arranged from left to right if all heads must be consecutive?

8. A social club consists of 10 pairs of twins, some identical, some fraternal. Of these 20 members, 13 are women and 7 are men.
   i. How many ways can this club select a social coordinator and treasurer if these roles must be served by different club members?
   ii. How many ways can this club select a social coordinator and treasurer if these roles must be served by club members of opposite gender?
   iii. How many ways can this club form a six person committee to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power.
   iv. How many ways can this club form a six person committee that consists of three twin pairs to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power.

v. How many ways can this club form a six person committee that consists of six club members but no twin pairs to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power.

9. Grimaldi Text Section 1.3: 2, 5, 6, 7, 8, 10, 13