Section 2.2: Axioms of Probability

1 Pop Quiz!

1. Dr. DeMaio's favorite sport on ESPN 8 is:
   a) Curling
   b) Dodgeball
   c) Lawn Mower Racing
   d) Caber Tossing
   e) Squirrel Water-Skiing

2. On college intramural teams, Dr. DeMaio's jersey number was always:
   a) π
   b) 13
   c) $\sqrt{-1}$
   d) $\lim\limits_{x \to 0} \frac{1}{x}$

2 Formal Probability

For the pop quiz questions above random guessing had to be employed. The probability of a correct answer was one out of the number of responses offered. This concept of counting and dividing is the heart of computing probabilities. What is the probability of a correct answer on the pop quiz? For #1, $p = \frac{1}{5} = .2 = 20\%$. For #2, $p = \frac{1}{4} = .25 = 25\%$.

Definition 1 *(no longer crude) Let an experiment consist of a collection of n disjoint and equally likely events. This collection is called the* ***sample space S****. Furthermore suppose exactly a of the events result in event A. Then the probability that event A will occur is $P(A) = \frac{a}{n}$. Additionally, the probability of the set of all possible outcomes of an experiment must be $P(S) = 1$.*

Probabilities must be between 0 and 1, inclusive. A probability of 0 indicates impossibility. A probability of 1 indicates certainty.

Example 1 *The probability that an 1107 student is both absent for class and present in class is 0.*

Example 2 *The probability that today is Monday and today is Tuesday is 0.*

Example 3 *When rolling a die with integer faces, the probability that the number will be even or odd is 1.*

Example 4 *The probability that someone will post something insensitive or offensive to the internet today is 1.*

Theorem 1 *$0 \leq P(A) \leq 1$.*

Proof. Clearly $n > 0$ and $a \geq 0$ so $P(A) = \frac{a}{n} \geq 0$. It is equally clear that $a \leq n$ so $P(A) = \frac{a}{n} \leq 1$. ■

Example 5 *Dr. DeMaio's morning MATH 1107 class contains 12 freshmen, 23 sophomores, 5 juniors and 11 seniors. If one student is selected at random, what is the probability that they are a senior? There are* $12 + 23 + 5 + 11 = 51$ *students in the class. Thus, the probability that a senior is selected is* $p = \frac{11}{51} = 0.215\,69 = 21.6\%$.

Theorem 2 *The **Law of Large Numbers** (LLN) says that the long-run relative frequency of repeated independent events gets closer and closer to the true relative frequency as the number of trials increases.*

For example, consider flipping a fair coin many, many times. The overall percentage of heads should settle down to about 50% as the number of outcomes increases.

The common (mis)understanding of the LLN is that random phenomena are supposed to compensate for whatever happened in the past. This is just not true. For example, when flipping a fair coin, if heads comes up on each of the first 10 flips, the probability of a tail on the next flip is still $p = \frac{1}{2}$. The coin does not remember what it did in the past. The coin does not feel bad that a tail has not been given a turn recently.

Thanks to the LLN, we know that relative frequencies settle down in the long run, so we can officially give the name probability to that value.

To compute a probability without running the experiment countless times to determine the true relative frequency of an event we consider all the equally likely possible outcomes of an experiment. It's equally likely to get any one of six outcomes from the roll of a fair die. It's equally likely to get heads or tails from the toss of a fair coin. However, keep in mind that events are not always equally likely. A skilled basketball player has a better than 50-50 chance of making a free throw. When rolling a pair of dice, a sum of seven and a sum of twelve are not equally likely events.

Example 6 *An experiment consists of randomly picking a card from a standard deck of playing cards (no jokers). Some possible outcomes of this experiment are listed below.*

A - The 8 of clubs is selected.
B - A red card is selected.
C - The Jack of hearts is not selected.

Problem 1 *Compte the probability of each of the following:*

1. A
2. B
3. C
4. $\bar{A}$
5. $\bar{B}$
6. $\bar{C}$
7. $A \cup B = A$ or B
8. $A \cap B = A$ and B
9. $A \cap C = A$ and C
10. $B \cup C = B$ or C

Problem 2 *When drawing a single card from a deck, what is the probability the card is a 4?*

Problem 3 *When drawing a single card from a deck, what is the probability the card is a club?*

Problem 4 *When drawing a single card from a deck, what is the probability the card is a 4 and a club?*

Problem 5 *When drawing a single card from a deck, what is the probability the card is red and a club?*

Problem 6 *When drawing a single card from a deck, what is the probability the card is not a club?*

Problem 7 *An experiment consist of flipping a fair coin twice. Compute the probabilities of the following events.*

A - Exactly one head is observed.
B - At least one head is observed.
C - No tails are observed.
The first step is to construct the sample space.

HH	TH
HT	TT

Problem 8 *Repeat for $n = 3$ flips of the coin*

Example 7 *A pair of fair dice is rolled. Compute the probabilities of the following events.*

A - The sum of the two dice is 7.
B - The sum of the two dice is 5
C - The sum of the two dice is an even number.

$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(6,1)$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	$(6,3)$
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	$(6,4)$
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	$(6,5)$
$(1,6)$	$(2,6)$	$(3,6)$	$(4,6)$	$(5,6)$	$(6,6)$

Let A be any event. Recall that the complement of A, $\bar{A}$ (also A^c or $A^{'}$) is the event that A does not occur.

Theorem 3 *For any event A, $P(\bar{A}) = 1 - P(A)$.*

Proof. Consider the collection of n disjoint and equally likely outcomes that form the sample space S and the a outcomes result in event A. Then the probability that event A will occur is $P(A) = \frac{a}{n}$. What is $P(\bar{A})$? If a outcomes satisfy A then $n-a$ outcomes satisfy $\bar{A}$. Thus, $P(\bar{A}) = \frac{n-a}{n} = \frac{n}{n} - \frac{a}{n} = 1 - \frac{a}{n} = 1 - P(A)$. ■

Theorem 4 *If A and B are disjoint (mutually exclusive) events then $P(A$ or $B) = P(A) + P(B)$.*

Proof. Consider the collection of n disjoint and equally likely outcomes that form the sample space S, the a outcomes result in event A and the b outcomes result in event B. Then the probability that outcome A or B occurs is $P(A$ or $B) = \frac{a+b}{n} = \frac{a}{n} + \frac{b}{n} = P(A) + P(B)$. ■

Example 8 *Pick a single card from a deck. What is the probability that you select an Ace or an 8?*
These are disjoint events. There are four of each rank in a deck of cards. Thus, the probability that you select an Ace or an 8 is $\frac{4}{52} + \frac{4}{52} = \frac{8}{52} = 0.153\,85$.

Exercise 1 *Pick a single card from a deck. What is the probability that you select a club or a diamond?*

Exercise 2 *Pick a single card from a deck. What is the probability that you select a six or a diamond? **Uh oh! These are not disjoint events!***

Theorem 5 *For any events A and B, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.*

Proof. *Consider the collection of n disjoint and equally likely outcomes that form the sample space S, the a outcomes result in event A, the b outcomes result in event B and the c events that are common to both A and B. Then the probability that outcome A or B occurs is $P(A \text{ or } B) = \frac{a+b-c}{n} = \frac{a}{n} + \frac{b}{n} - \frac{c}{n} = P(A) + P(B) - P(A \text{ and } B)$.* ■

Exercise 3 *Pick a single card from a deck. What is the probability that you select a six or a diamond? $P(6 \text{ or } D) = P(6) + P(D) - P(6 \text{ and } D) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13} = 0.307\,69$.*

Exercise 4 *Pick a single card from a deck. What is the probability that you select a nine or a club?*

Exercise 5 *A study at a local bar found people of various ages playing games. It is a strange bar in that everyone is engaged in exactly one activity; no more, no less*

	21-29	30-39	40-49	50 and older	Total
Darts	4	12	15	6	37
Pool	8	17	16	11	52
Karaoke	17	5	0	1	23
Total	29	34	31	18	112

Find the probability that a randomly selected person...

1. is playing pool;

2. is 30-39;

3. is playing pool or darts;

4. is 30-39 or throwing darts;

5. is playing pool or 50 and older;

Problem 9 *Construct a formula for* $P(A \cup B \cup C)$.

Remark 6 *This process can continue indefinately. Here we've covered the first two cases of what is known as the Principle of Inclusion and Exclusion.*

Exercise 6 *Pick a single card from a deck. What is the probability that your card is red or a club or a nine?*
Use the formula derived above for $P(A \cup B \cup C)$.

Use complements.