Section 2.3 Counting Techniques

1 Multiplication Rule

A basic counting technique in combinatorics is the multiplication rule. Consider a club that consists of three individuals: Mark, Kate and Pat. Every club needs a president and a vice-president, so the question becomes how many different elected choices of president and vice-president can be chosen for this small club? In this case, it is very easy to write down all possible selections as demonstrated to the left.

Pres.	V-P
Mark	Kate
Mark	Pat
Kate	Mark
Kate	Pat
Pat	Mark
Pat	Kate

Clearly, there are six different ways to choose a president and vice-president. But, what if the club had twenty members? This brute force method would not be a reasonable approach. It would be difficult to be certain that all election possibilities had been found. There is a need for a more sophisticated counting technique rather than just primitive brute force.

Upon closer analysis, it is clear that there are three people who may be selected as president. For each choice of president, there are then two choices for vice-president. It is no coincidence that $3 * 2 = 6$. If the club consisted of 20 people, there would be $20 * 19 = 380$ ways of choosing the president and vice-president. This is the fundamental idea behind the most important theorem in combinatorics.

Theorem 1 ***Multiplication*** *(or* ***Product****) Rule: If task one can be performed in n_1 ways and task two can be performed in n_2 ways regardless of how task one was performed, then the total number of ways of performing task one and then task two is $n_1 n_2$.*

Example 2 *As an example, suppose a state only needs 12 million unique license plates. Will enough different license plates exist with three uppercase letters followed by three digits? There are six different tasks. The first three tasks are to pick the first, second and third letters. When deciding on the number of ways that the second letter can be selected, the question of repetition arises. Can the second letter be identical to the first? In this case, it is common for license plates to contain multiple copies of the same letter or digit. Hence, repetition is allowed. The last three tasks are to pick the first, second and third digits. The multiplication rule indicates that $26^3 10^3 = 17\,576\,000$ different license plates exist which is sufficient for the state's needs.*

Example 3 *Suppose this state also wishes to not repeat characters in its license plates. Will enough different license plates exist using the scheme of the previous problem but disallowing repetition? Now, the second letter selected must be different from the first letter selected and regardless of the specific letter there are only 25 letters available. The tasks are remain the same but now result in $26 * 25 * 24 * 10 * 9 * 8 = 11\,232\,000$ different license plates. Only a small change in design was made but but the number of plates was reduced over 33% and the supply of license plates is now insufficient.*

Remark 4 *The careful reader of Theorem 1 is now asking what the phrase, regardless of how the first task is performed, means. It means the multiplication rule holds only if regardless of how the first task is performed, there are always the same number of choices for performing the second task. This concept is illustrated in the above example. In some problems the counting technique may have to be changed to achieve this. The following example illustrates how the multiplication rule may be applied in different situations by choosing appropriate tasks.*

Example 5 *Consider the twenty person club made up of eight men and twelve women. Suppose a president and vice-president of opposite gender must be selected. There are still twenty choices for president but a sticky point arises when we go to choose a vice-president. On one hand, if a male president is selected then there are twelve choices for a female vice-president. On the other hand, if a female president is chosen then there are eight choices for a male vice-president. In this case, the number of ways task two can be performed changes based on the choice for task one. In order to use the multiplication rule here, the problem must be broken into three tasks rather than two. Task one is to pick a male to serve in office. Task two is to pick a female to serve in office. Finally, task three selects the gender of the president (which automatically selects the other gender to serve as vice-president). Hence, the number of different ways to select a president and vice-president of opposite gender is* $8 * 12 * 2 = 192$*. While the multiplication rule did not work as easily as we might have preferred, ultimately the rule was sufficient for the problem at hand.*

Problem 6 *How many different passwords exist using four lowercase letters followed by two digits (0-9)?*

Problem 7 *How many different passwords exist using four distinct lowercase letters followed by two distinct digits (0-9)?*

Problem 8 *How many different passwords exist using four distinct lowercase or uppercase letters followed by two distinct digits (0-9)?*

Problem 9 *How many different passwords of seven characters exist where each character may be a lowercase letter of the alphabet or a digit?*

Problem 10 *How many different passwords of seven characters exist where each character may be a lowercase or uppercase letter of the alphabet or a digit?*

Problem 11 *How many different ways can three six-sided dice be rolled?*

Problem 12 *How many six digit numbers have no two consecutive digits the same? Think about the difference between a 6 digit number and a 6 digit password.*

2 Exercises

1. A collection of seven distinct coins will be arranged from left to right. There are four heads face up and three tails face up.
i. How many different ways can the coins be arranged from left to right?
ii. How many different ways can the coins be arranged from left to right if there can be no consecutive heads?
iii. How many different ways can the coins be arranged from left to right if all heads must be consecutive and all tails must be consecutive?
iv. How many different ways can the coins be arranged from left to right if all heads must be consecutive?

2. Two pairs of siblings (two brothers and a brother/sister), and three only children (two men and one woman) sit in a row of seven consecutive seats. How many ways can they be seated:
i. with no restrictions;
ii. alternating genders;
iii. such that the women are all consecutive;
iv. such that twin siblings sit next to one another?

3 Multiplication Rule: Subtraction and Factorials

We now introduce some basic terminology. Suppose you wanted to count how many different batting orders exist for the nine positions for an American League baseball team where any position can bat at any spot in the order. This is a direct application of the multiplication rule. There are nine tasks to perform. Pick a position to bat first and then a position to bat second, etc. Clearly there are $9*8*7*6*5*4*3*2*1 = 362\,880$ ways to do this. Each of the 362,880 different orders of these nine positions is called a permutation. The product of all the integers from 1 to k is denoted as k-factorial. The total number of permutations of k objects is k-factorial. Rather than write the product of k integers every time a factorial is needed, an exclamation mark is used to denote the factorial operation.

Definition 13 *The **factorial** function $k! = k(k-1)(k-2)\cdots 3*2*1$ for every positive integer k. Furthermore, as a matter of convenience, we define $0! = 1$. Note that $k!$ grows very quickly as the table illustrates.*

$0! = 1$
$1! = 1$
$2! = 2$
$3! = 6$
$4! = 24$
$5! = 120$
$6! = 720$
$7! = 5040$
$8! = 40\,320$
$9! = 362\,880$
$10! = 3628\,800$

Example 14 *How many different permutations of the characters $\{a, b, c, d\}$ exist. There are four characters so there will be different $4! = 24$ permutations. All possible permutations are listed to the right.*

abcd	abdc	acdb
acbd	adbc	adcb
bacd	badc	bcad
bcda	bdac	bdca
cabd	cadb	cbad
cbda	cdab	cdba
dabc	dacb	dbac
dbca	dcab	dcba

At times we may not need to order all elements in a set. It is still easy to factorial notation to represent such a count.

Example 15 *How many different permutations of two characters from* $\{a, b, c, d, e\}$ *exist. We can pick these two ordered elements in* $5 * 4 = \frac{5!}{(5-2)!} = \frac{5!}{3!}$ *ways.*

Definition 16 *The number of permutations of* k *objects from a set of* n *objects is* ${}_nP_k = \frac{n!}{(n-k)!}$.

Definition 17 *Veteran wisdom in mathematics states that a technique is something you use over and over again while a trick is something you use once. Students claim any technique that differs slightly from the example is a trick. I'm certain the truth lies somewhere in the middle. Initial problems utilizing the multiplication rule can seem trivial. There are some very standard ways to use the multiplication rule that are not obvious at first glance.*

Another approach to the president and vice-president of opposite gender problem would be to count all the possible ways to select a president and vice-president without regard to gender and then subtract off all the undesirable selections. There are $20*19 = 380$ ways to select a president and vice-president. The unwanted selections are those selections from the same gender. There are $8*7 = 56$ ways to select a male president and male vice-president. There are $12*11 = 132$ ways to select a female president and female vice-president. Thus, there are $380 - 56 - 132 = 192$ different ways to select a president and vice-president of opposite gender. Counting all possible cases and then subtracting of forbidden cases is a very useful technique. This technique works exceptionally well when encountering problems involving the form "at least" or "at most."

In our twenty person club, we have eight seniors, four juniors, three sophomores and five freshmen. How man ways can we select a President, Vice-President and Treasurer with at least one senior serving in office? We have $20 * 19 * 18 = 6840$ ways to select these officers. However, some of these selections do not have the required senior among them. How many? There are $12*11*10 = 1320$ selections without a senior. Hence, there are $6840 - 1320 = 5520$ with at least one senior.

4 Exercises

1. Dr. Ian Malcolm is arranging his favorite books about dinosaurs (*Jurassic Park*, *God Creates Dinosaurs, The Lost World of the Dinosaurs* and *Dinosaur Detective*) on his bookshelf. In how many ways can Ian arrange his books from left to right?

2. In an attempt to raise productivity the CANE corporation is scheduled to publicly flog its six least productive employees. In how many different orders can these six employees be made an example of?

3. How many different ways can the letters in the word **riot** be arranged? Find them all.

4. How many different ways can the letters in the word **stygian** be arranged? How many different ways can the letters in the word **stygian** be arranged such that the word **sing** appears in the arrangement?

5. How many different passwords of seven characters exist where each character may be a lowercase letter of the alphabet and contains at least one vowel?

6. Two pairs of siblings (two brothers and a brother/sister) and three only children (two men and one woman) sit in a row of seven consecutive seats. How many ways can they be seated with no siblings beside each other?

7. True or False? $(n+k)! = n! + k!$ for all positive integers n and k. If true, prove the statement. If false, provide a counterexample to demonstrate that the statement is false.

8. True or False? $(n+k)! = n! * k!$ for all positive integers n and k. If true, prove the statement. If false, provide a counterexample to demonstrate that the statement is false.

9. True or False? $(nk)! = n! * k!$ for all positive integers n and k. If true, prove the statement. If false, provide a counterexample to demonstrate that the statement is false.

10. Prove $\frac{n!}{(n-2)!2!}$ is an integer for all integers $n \geq 2$.

5 Unordered Selections without Repetition

All the examples presented so far have an inherent order implied in the statement of the problem. Virtually every application of the multiplication rule introduces order into a problem, usually in a subtle way. Consider the following two examples where order is not important in the problem.

Example 18 *Two friends are planning to order pizza for dinner. Toppings offered by the restaurant are mushrooms, sausage, pepperoni, anchovies, onions and peppers. How many different combinations of toppings can be ordered for the pizza? There are six potential toppings and for each the only decision is to include the topping on the pizza or not to include the topping on the pizza. This breaks down to six tasks to perform where each task can be performed in two ways. There exist* $2^6 = 64$ *different ways to choose toppings for this pizza. This runs the gamut of a plain pizza (no toppings) to "all the way."*

One of the first steps in solving counting problems is to determine whether or not order is important. And while almost all problems where order is important involve the multiplication rule, there are solutions to problems where order is not important that also employ this rule. Whenever the multiplication rule is used it is crucial to understand the role order is playing in the solution.

Example 19 *Consider the pizza problem again. There are still six toppings-mushrooms, sausage, pepperoni, anchovies, onions and peppers- from which to choose. Now the task is to determine how many pizzas exist with exactly three toppings. The naive approach uses three tasks, pick topping number one, pick a different topping number two and pick a different topping number three. This would be done in ways* $6*5*4=120$*. However, closer analysis is needed.*

topping one	topping two	topping three
onions	peppers	sausage
onions	sausage	peppers
peppers	onions	sausage
peppers	sausage	onions
sausage	onions	peppers
sausage	peppers	onions

These six cases are considered different in the 120 selections because the design of the three tasks implies that peppers as topping one is somehow different from peppers as topping two or as topping three. The multiplication rule has introduced an order to the toppings when no order is required. As any experienced pizza connoisseur will tell you, the order that the toppings go onto the pizza is irrelevant! All that matters is whether the topping is present or absent. One solution to fix this problem is to eliminate the repetition. Each set of three toppings will give rise to $3! = 6$ permutations as the example above does. Thus, the true number of three topping pizzas is $\frac{6*5*4}{3!} = 20$. This example demonstrates the need for a method for picking objects as a set without forcing an ordering of the elements.

The general question we ask is, how many ways can k objects be selected from n objects as a subset with no implied order? Using the above strategy, first k objects are selected. This can be done in $n(n-1)\cdots(n-k+1)$ ways. Second, the repetition must be eliminated. Each set of k objects occurs in $k!$ different orders. The total number of ways of selecting a subset of k objects from n objects is $\frac{n(n-1)\cdots(n-k+1)}{k!}$. With a little manipulation this can be written as $\frac{n!}{k!(n-k)!}$ and the result of Theorem 20 follows.

Theorem 20 *The number of ways of choosing* k *distinct objects as a subset from* n *distinct objects is* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ *for integers* $0 \le k \le n$*. The symbol is called the binomial coefficient, and is read "n choose k".*

The binomial coefficient $\binom{n}{k}$ (also commonly written as $C_{n,k}$ or ${}_nC_k$) is the number of combinations of n items taken k at a time. With this notation, the solution to the above pizza problem is $\binom{6}{3} = 20$.

The values of five particular pairs of parameters are simple to determine based solely on the fact that the binomial coefficient counts the number of ways to select k items from n items as a subset. We will establish these values without resorting to the formula of Theorem 20. Rather we focus on the logic of the selections. For all positive integers n, $\binom{n}{0} = 1$ because there is only one way to select the empty set. Likewise, $\binom{n}{n} = 1$ as there is only one way to select the set itself. The binomial coefficient $\binom{n}{1} = n$ since there are n different options for a single selection. Likewise $\binom{n}{n-1} = n$ as there are n different choices for the item to not be selected. Finally, $\binom{n}{k} = 0$ for $k > n$ since one cannot select more items than are available without repetition.

Example 21 *Returning to the twenty person club from the previous section, how many ways can a three person committee be selected if all three individuals are of equal rank and power? This is enumerated by the binomial coefficient twenty choose three since the order in which the three are selected is not important. This results in* $\binom{20}{3} = 1140$ *possible committees.*

Example 22 *How many subsets of size two from* $S = \{a, b, c, d, e, f, g, h\}$ *exist? Just as in the pizza problem an unordered selection of two elements is required. This selection can be made in* $\binom{8}{2} = 28$ *ways.*

Example 23 *Even though the binomial coefficient easily counts unordered combinations, sometimes a different perspective may aid in a solution. Bethany and Tyler will each buy five different pieces of candy from twelve different types of candy. Bethany selects her five pieces by selecting her five favorite types. Bethany does this in* $\binom{12}{5} = 792$ *different ways. Tyler is somewhat negative and reasons that he will select his five pieces of candy by removing his seven least favorite kinds of candy. The five remaining candies will then be his selection. Tyler can do this in* $\binom{12}{7} = 792$ *different ways. It is no coincidence that* $\binom{12}{5} = \binom{12}{7}$*. Designating* k *objects to be selected from* n *is actually the same as designating* $n - k$ *objects from* n *to not be selected. It will always be the case that* $\binom{n}{k} = \binom{n}{n-k}$*, the proof of which is left for the reader as Problem* **??***.*

Example 24 *How many different arrangements of the letters in the word moon exist? The answer is not* $4!$ *since the letters are not all distinct. One solution is to divide out the repetition caused by the duplicate letters. This approach yields* $\frac{4!}{2!} = 12$ *arrangements. Alternatively, we could first select two locations for the ohs in an unordered fashion and then permute the remaining distinct letters for a total of* $\binom{4}{2} * 2! = 12$ *arrangements.*

mnoo
mono
omno
moon
omon
oomn
nmoo
nomo
onmo
noom
onom
oonm

Example 25 *How many three digit numbers exist such that each digit is larger than the digit to its right? This problem seems to be tailor-made for the multiplication rule where order is important. Looks can be deceiving. First, note that the digits must be distinct since no digit is larger than itself. Second, given any three different digits, there exists exactly one way to arrange them so that each digit is larger than the digit to its right. If the digit set* $\{3, 8, 9\}$ *is selected then only the number 983 can be constructed to satisfy the given restrictions. The original problem reduces to an unordered selection of three distinct digits from 0 through 9. This can be done in* $\binom{10}{3} = 120$ *ways. What first appeared to be a problem where order is important turns out not to be.*

Theorem 26 *For integers n and k,* $\binom{n}{k} = \binom{n}{n-k}$.
Proof. *The proof begins by applying the formula for binomial coefficients to the right-hand side of the equation and simplifying.* ■

$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$

Theorem 27 *For integers n and k,* $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
Proof. *Our proof begins by applying the formula for binomial coefficients to the right-hand side of the equation and simplifying. To add the two resultant fractions, a common denominator is found. Adding and simplifying the results finishes the proof.*

$$\begin{aligned} & \binom{n-1}{k} + \binom{n-1}{k-1} \\ = & \frac{(n-1)!}{k!\,(n-k-1)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!} \\ = & \frac{(n-k)\,(n-1)! + k\,(n-1)!}{k!\,(n-k)!} \\ = & \frac{n!}{k!\,(n-k)!} \\ = & \binom{n}{k} \end{aligned}$$

■

6 Exercises

1. In a forty person club, how many different ways can a
   i. president, vice-president, and secretary be selected;
   ii. three person committee be chosen?

2. In our twenty person club, we have eight seniors, four juniors, three sophomores and five freshmen. How man ways can we select a five person committee with at least one senior?

3. A restaurant offers pizza with eight different toppings available. If a large, medium or small pizza with any combination of toppings (but no toppings may be repeated, i.e. no double pepperoni) and any number of toppings can be selected then how many different pizzas with exactly four toppings can be ordered?

4. How many different ways can the letters in the following words be arranged?
   i. house
   ii. building
   iii. construction
   iv. structures

5. In the game of Clue, there are six suspects (Col. Mustard, Prof. Plum, Mr. Green, Mrs. Peacock, Miss Scarlet and Mrs. White), six possible weapons (Knife, Candlestick, Revolver, Rope, Lead Pipe and Wrench) and nine locations (Hall, Lounge, Dining Room, Kitchen, Ball Room, Conservatory, Billiard Room, Library and Study). The murder of Mr. Boddy was committed by one suspect, with one weapon in one location.
   i. How many different possible ways could the murder have been committed?
   ii. How many ways could Miss Scarlet have committed the murder?

6. Jason's CD collection consists of five different rock CD's, three different jazz CD's, two different blues CD's, two different classical CD's and a single folk CD. Jason is planning a trip and randomly selects four CD's.
   i. How many different ways can this be done?
   ii. How many different ways can this be done if the folk CD must be one of the four?
   iii. How many different ways can this be done if Jason will take exactly one rock CD?
   iv. How many different ways can this be done if Jason will take at least one rock CD?

7. A collection of seven indistinguishable coins will be arranged from left to right. There are four heads face up and three tails face up.
   i. How many different ways can the coins be arranged from left to right?
   ii. How many different ways can the coins be arranged from left to right if there can be no consecutive heads?
   iii. How many different ways can the coins be arranged from left to right if all heads must be consecutive and all tails must be consecutive?
   iv. How many different ways can the coins be arranged from left to right if all heads must be consecutive?

8. A social club consists of 10 pairs of twins, some identical, some fraternal. Of these 20 members, 13 are women and 7 are men.
   i. How many ways can this club select a social coordinator and treasurer if these roles must be served by different club members?
   ii. How many ways can this club select a social coordinator and treasurer if these roles must be served by club members of opposite gender?
   iii. How many ways can this club form a six person committee to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power.
   iv. How many ways can this club form a six person committee that consists of three twin pairs to coordinate the club trip to the annual national meeting of twins? Note that each person on this committee is of equal rank and power.
   v. How many ways can this club form a six person committee that consists of six club members but no twin pairs to coordinate the club trip

to the annual national meeting of twins? Note that each person on this committee is of equal rank and power.

9. Prove $\binom{2n}{2} = n^2 + 2\binom{n}{2}$ for all positive integers n.