Section 3.1, 3.2: Discrete Random Variables

1 Two Examples

Definition 1 *A **random variable** is a function that assigns a numerical value to each outcome in a sample space.*

There are two types of random variables, discrete and continuous, based on the data types assigned to values. The above two examples illustrate the different ways a random variable must be defined. Accordingly, different techniques will be needed to work with the two different types of random variables.

Example 2 *Consider the experiment of rolling a pair of dice and summing the faces. The random variable X assigns to each roll its sum. The following table indicates the probabilities for each value in X.*

Sum	2	3	4	5	6	7	8	9	10	11	12

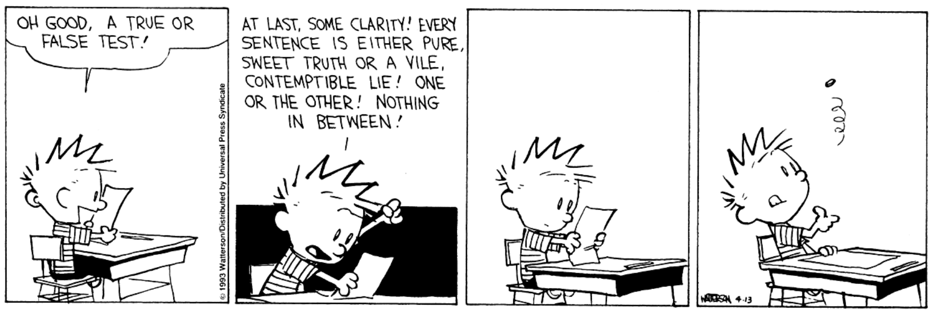
Remark 3 *Since the data is discrete, we call this function a **discrete random variable**.*

Example 4 *Let Y be the random variable that assigns to each human being, their height in inches. Given the uncountably infinite number of possible heights, it is not possible to define Y based on a table.*

Remark 5 *Since the data is continuous, we call this function a **continuous random variable**.*

Problem 6 *Consider the experiment of rolling a three sided-die and a 4-sided die and summing the faces. Construct a discrete random variable X.*

Definition 7 *A random variable with only two outcomes (0-1, true-false, right-wrong, on-off, etc.) is called a* ***Bernoulli random variable****.*



Example 8 *A multiple choice test with only one correct answer per questions is a Bernoulli random variable.*

Problem 9 *Formally describe a public safety officer's parking lot duty as a random variable.*

Section 3.3: Probability Distributions for Discrete Random Variables

Definition 10 *A* ***probability density function (pdf) or probability mass function (pmf)*** *for a discrete random variable X is a function whose domain is all possible values of X and assigns to each $x \in X$ the probability that x occurs. Note that the sum of all probabilities in a distribution must be 1.*

Problem 11 *Is the following function a probability distribution? Explain.*

X	1	2	3
$P(X)$	0.5	0.6	-0.1

Example 12 *Consider the following probability distribution.*

X	1	2	3	4	5	6
$P(X)$	0.3	0.2	0.1	0.1		0.2

1. $P(X = 3) =$
2. $P(X = 4) =$
3. $P(X = 5) =$
4. $P(X \leq 3) =$
5. $P(X < 3) =$
6. $P(2 \leq X \leq 4) =$
7. $P(X > 5) =$

Example 13 *Consider the experiment of rolling a pair of dice and summing the faces. The random variable X assigns to each roll its sum. The following table indicates the probabilities for each value in X. Together they form the probability density function.*

Sum	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Problem 14 *Consider the experiment of rolling a three sided-die and a 4-sided die and summing the faces. Construct the pdf for X.*

Problem 15 *A $2 lottery ticket offers four chances to win different amounts of money as indicated by the following probability distribution model.*

Prize	$0	$0.5	$1	$5	$50
Prob	$\frac{71}{100}$	$\frac{15}{100}$	$\frac{10}{100}$	$\frac{3}{100}$	$\frac{1}{100}$

1. What is the probability that a ticket wins nothing?

2. What is the probability that a ticket wins money but is a losing proposition overall?

3. Two people buy one ticket each. What is the probability that they both win nothing?

4. Two people buy one ticket each. What is the probability that they both win money?

5. Ten people buy one ticket each. What is the probability that at least one person wins money?

6. Five hundred people buy one ticket each. What is the probability that at least one person wins the $50 prize?

Problem 16 *You draw a card from a deck. If you get a club you get $5. If you get an Ace you get $10. For all other cards you receive nothing. Let X be the random variable of money won. Create a probability distribution model for this game.*

Problem 17 *You draw a card from a deck. If you get a club you get nothing. If you get red card you get $10. If you get a spade you get $15 and get to select another card (without replacement). If the second is another spade you receive an additional $20. You receive nothing for any non-spade card. Create a probability model for this game.*

A random variable with a finite number of outcomes is necessarily discrete. However, it is possible for a discrete random variable to have an infinite number of outcomes.

Example 18 *John flips a coin until he observes a tail. Construct the pdf for the number of flips of the coin for this discrete random variable.*

2 Cumulative Distribution Functions

The cumulative distribution function of a discrete pdf is the sum of all probabilities of outcomes less than or equal to some fixed value x.

Definition 19 *The cumulative distribution function $F(x)$ of a pdf $p(x)$ is* $\sum_{y \leq x} p(y)$.

Example 20 *Consider the experiment of rolling a pair of dice and summing the faces. The random variable X assigns to each roll its sum. The following table indicates the probabilities for each value in X. Together they form the probability density function.*

Sum	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Now construct the cumulative distribution function of this pdf.

Sum	2	3	4	5	6	7	8	9	10	11	12
$F(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36}$

Problem 21 *Consider the experiment of rolling a three sided-die and a 4-sided die and summing the faces. Construct $p(x)$ and $F(x)$ for X.*

Example 22 *Consider rolling a fair die until a 5 appears. Construct the pdf for the number of rolls.*
The probability of rolling a 5 (or any specific value) on a fair die is $p = \frac{1}{6}$. *Since there is never a guarantee of ever getting a 5, the set of possible values for* x *is the set of positive integers. Since* Z^+ *is infinite, one cannot write down all possible values and probabilities. Instead we need to construct a function. It can help to model a few specific values of* x *before writing a general form. For example, the first 5 occurs on attempt number four with probability* $\left(\frac{5}{6}\right)^3 * \frac{1}{6} = \frac{125}{1296}$. *One has to roll a number other than 5, three times and then roll a 5. Generalizing this our pdf,* $p(x) = \left(\frac{5}{6}\right)^{x-1} * \frac{1}{6}$.

Example 23 *Consider rolling a fair die until a 5 appears. Construct the cumulative density function for the number of rolls. Since* Z^+ *is infinite we once again need to construct a general formula.*

$$\begin{aligned} F(X) &= \sum_{y=1}^{x} p(y) \\ &= \sum_{y=1}^{x} \left(\frac{5}{6}\right)^{y-1} * \frac{1}{6} \\ &= \frac{1}{6} \sum_{y=1}^{x} \left(\frac{5}{6}\right)^{y-1} \\ &= \frac{1}{6} \sum_{y=0}^{x-1} \left(\frac{5}{6}\right)^{y}. \end{aligned}$$

Fortunately this is a geometric series which converges nicely.

Remark 24 *For those who need a slight refresher on the geometric series, let* $s = a + ar + ar^2 + ... + ar^{n-1}$. *Hence*

$$\begin{aligned} rs &= r\left(a + ar + ar^2 + ... + ar^{n-1}\right) \\ &= ar + ar^2 + ... + ar^n \end{aligned}$$

and now

$$\begin{aligned} s - rs &= \\ & \quad a + ar + ar^2 + ... + ar^{n-1} \\ & \quad - \left(ar + ar^2 + ... + ar^{n-1} + ar^n\right) \\ &= a - ar^n \\ & \quad \textit{which implies} \\ s(1-r) &= a(1-r^n) \\ & \quad \textit{which implies} \\ s &= \frac{a(1-r^n)}{(1-r)}. \end{aligned}$$

Returning back to

$$
\begin{aligned}
F(x) &= \frac{1}{6}\sum_{y=0}^{x-1}\left(\frac{5}{6}\right)^{y} \\
&= \frac{1}{6} * \frac{a(1-r^{n})}{(1-r)} \\
\text{where } a &= 1\text{and } r = \frac{5}{6} \\
&= \frac{1}{6} * \frac{(1-\left(\frac{5}{6}\right)^{n})}{(1-\frac{5}{6})} \\
&= \frac{1}{6} * \frac{(1-\left(\frac{5}{6}\right)^{n})}{(\frac{1}{6})} \\
&= (1-\left(\frac{5}{6}\right)^{n}).
\end{aligned}
$$