Section 3.4: Expected Value (Mean) and Standard Deviation for a Discrete Random Variable

Recall the experiment of rolling a pair of dice and summing the faces. The random variable X assigns to each roll its sum. The following table indicates the probabilities for each value in X. Together they form the probability density function.

Sum	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The pdf provides an easy table or function from which to acquire probabilities of individual roles. What value on average do we expect after rolling the pair of dice? What value on average do we expect after rolling the pair of dice 100 times? How much variation do we expect from that average roll? These sound like questions of a mean and standard deviation. Rather than being a mean and standard deviation of a data set, the pdf introduces the notion of uncertainty and likelihood.

Definition 1 *The* ***expected value*** *(or* ***mean****), $E(X)$(or μ), of a discrete probability distribution is given by*

$$E(X) = \sum_{x \in X} x * p(x).$$

Definition 2 *The* ***variance,*** *$V(X)$, of a discrete probability distribution is given by*

$$\sigma^2 = \sum_{x \in X} (x - E(X))^2 * p(x).$$

Example 3 *Back to the experiment of rolling a pair of dice and summing the faces. The random variable X assigns to each roll its sum. The following table is our previously constructed probability model.*

Sum	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

What is the expected value of the sum of a pair of dice?

$$\begin{aligned} E(X) &= \sum_{x \in X} x * p(x) \\ &= 2 * \frac{1}{36} + 3 * \frac{2}{36} + 4 * \frac{3}{36} + 5 * \frac{4}{36} + 6 * \frac{5}{36} + 7 * \frac{6}{36} \\ &\quad + 8 * \frac{5}{36} + 9 * \frac{4}{36} + 10 * \frac{3}{36} + 11 * \frac{2}{36} + 12 * \frac{1}{36} \\ &= \frac{2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12}{36} \\ &= 7. \end{aligned}$$

What is the standard deviation of the sum of a pair of dice?

Sum	2	3	4	5	6	7	8	9	10	11	12
$(x - E(X))^2$	25	16	9	4	1	0	1	4	9	16	25
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\begin{aligned} V(X) &= \sigma^2 = \sum_{x \in X} (x - E(X))^2 * p(x) \\ &= 25 * \frac{1}{36} + 16 * \frac{2}{36} + 9 * \frac{3}{36} + 4 * \frac{4}{36} + 1 * \frac{5}{36} + 0 * \frac{6}{36} \\ &\quad +1 * \frac{5}{36} + 4 * \frac{4}{36} + 9 * \frac{3}{36} + 16 * \frac{2}{36} + 25 * \frac{1}{36} \\ &= \frac{35}{6}. \end{aligned}$$

Thus, $\sigma = \sqrt{\frac{35}{6}} = 2.4152$.

Example 4 *What is the probability that the sum of two dice falls within one standard deviation of the mean?*
One standard deviation from the mean is $(7 \pm 2.4152) = (4.5848, 9.4152)$. *So we could see a sum of 5, 6, 7, 8, or 9. This occurs with probability*

$$p = \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} = \frac{2}{3} = 0.66667.$$

Exercise 5 *A \$2 lottery ticket offers four chances to win different amounts of money as indicated by the following probability distribution model. Determine the expected value of buying a single ticket. Do you wish to play this game?*

Prize	\$0	\$0.5	\$1	\$5	\$50
Prob	$\frac{71}{100}$	$\frac{15}{100}$	$\frac{10}{100}$	$\frac{3}{100}$	$\frac{1}{100}$
Value	-2	-1.5	-1	3	48

So,

$$\begin{aligned} E(X) &= \sum_{x \in X} x * p(x) \\ &= -2 * \frac{71}{100} - 1.5 * \frac{15}{100} - 1 * \frac{10}{100} + 3 * \frac{3}{100} + 48 * \frac{1}{100} \\ &= -1.175. \end{aligned}$$

I personally would not want to play this game with a negative expected value and such a small prize.

Exercise 6 *What is the expected value of buying 50 such lottery tickets?*
$-1.175 * 50 = -58.75$.

Problem 7 *You pay \$1 to play a game. The game consists of rolling a pair of dice. If you observe a sum of 7 or 11 you receive \$4. If not, you receive nothing. Compute the expected value for this game?*

Example 8 *Consider flipping an unbalanced coin that lands H 60% of the time. What is the expected value and standard deviation for this Bernoulli random variable?*

Coin	*H*	*T*
Prob	.6	.4

is the pdf. It's expected value is that the coin lands H, 60% of the time. One can view this representing a success with a 1 and a failure as a 0 for the X values.

X	1	0
$p(x)$	.6	.4

$$\begin{aligned} E(X) &= \sum_{x \in X} x * p(x) \\ &= 1 * .6 + 0 * .4 \\ &= .6 \end{aligned}$$

$$\begin{aligned} \sigma^2 &= \sum_{x \in X} (x - E(X))^2 * p(x) \\ &= (1 - .6)^2 * .6 + (0 - .6)^2 * .4 \\ &= .24 \\ \textit{Thus, } \sigma &= \sqrt{.24} \\ &= 0.489\,90 \end{aligned}$$

While not immediately obvious, it should be noted that $\sigma^2 = .6 * .4$.

Example 9 *What is the expected value and standard deviation for a Bernoulli random variable with probability of success p?*

X	1	0
$p(x)$	p	$1 - p$

$$\begin{aligned} E(X) &= \sum_{x \in X} x * p(x) \\ &= 1 * p + 0 * (1 - p) \\ &= p \end{aligned}$$

$$
\begin{aligned}
\sigma^2 &= \sum_{x\in X}(x-E(X))^2 * p(x) \\
&= (1-p)^2 * p + (0-p)^2 * (1-p) \\
&= (1-p)(1-p)p + p^2(1-p) \\
&= (1-p)p[(1-p)+p] \\
&= p(1-p) \\
\text{Thus, } \sigma &= \sqrt{p(1-p)}
\end{aligned}
$$

Example 10 *It estimated 35% of students in the MSAS program work during the day. What is the standard deviation for this Bernoulli random variable?* $\sigma = \sqrt{p(1-p)} = \sqrt{.35(1-.35)} = 0.476\,97$.

At times we will want to use a given expectation as an input variable into some other function.

Example 11 *Our electronics store buys 5 game consoles at $250 each. The pdf for the number of consoles sold in the first week is given below.*

# sold	0	1	2	3	4	5
p(x)	.1	.1	.2	.3	.2	.1

We expect to sell $0*.1+1*.1+2*.2+3*.3+4*.2+5*.1 = 2.7$ *consoles this week.*
This week the console is hot and can be sold for $350 each. After this week, we must sell the remaining consoles at $200 each. What is our expected profit? If x *represents the number of consoles sold then our profit function is* $f(x) = 100*x - 50*(5-x) = 150x - 250$. *Our expected profit is*

$$
\begin{aligned}
E(f(x)) &= \sum_{i=0}^{5} f(i) * p(i) \\
&= f(0)*p(0) + f(1)*p(1) + f(2)*p(2) \\
&\quad +f(3)*p(3) + f(4)*p(4) + f(5)*p(5) \\
&= -250*.1 - 100*.1 + 50*.2 + 200*.3 + 350*.2 + 500*.1 \\
&= 155.
\end{aligned}
$$

Remark 12 *Note with wonder and amazement that* $150*2.7-250 = 155.0$.

Theorem 13 *For the expected value of linear function* $h(x) = ax+b$, $E(aX+b) = a*E(X)+b$.

Proof. $E(aX+b) = \sum[(ax+b)*p(x)] = \sum[ax*p(x)+b*p(x)] = \sum[ax*p(x)] + \sum(b*p(x)) = a\sum[x*p(x)] + b = a*E(X)+b$. ■

Example 14 *Assume the price on the hot console changes to \$425 in the first week and to \$175 after the first week while the likelihoods of selling x number of consoles remains the same. Now, what is the expected profit?*
Our profit function $f(x) = 175 * x - 75 * (5 - x) = 250x - 375$. *Now, expected value is* $250 * 2.7 - 375 = 300.0$.

Remark 15 *It is also true that* $V(aX+b) = a^2V(X)$ *and thus,* $\sigma_{aX+b} = |a|\sigma_X$. *Why do these identities make sense based on the meaning of an average and standard deviation?*

What other properties of expected value and standard deviation exist?

Theorem 16 *(Shortcut formula for* σ^2*) Variance of the random variable X ,*$\sigma^2 = E(X^2) - [E(X)]^2$.

Proof. $\sigma^2 = \sum_{x \in X} (x - E(X))^2 * p(x)$ from the definition of standard deviation. Rather than evaluate $x - E(X)$ for a value, algebraically square it. Bear in mind that $E(X)$ is a constant that can be pulled across a sum as well as constant factors.

$$\begin{aligned}
\sigma^2 &= \sum_{x \in X} (x - E(X))^2 * p(x) \\
&= \sum_{x \in X} \left(x^2 - 2xE(X) + E(X)^2\right) * p(x) \\
&= \sum_{x \in X} x^2 p(x) - \sum_{x \in X} 2xE(X)p(x) + \sum_{x \in X} E(X)^2 * p(x) \\
&= E(X^2) - 2E(X) \sum_{x \in X} xp(x) + E(X)^2 \sum_{x \in X} *p(x) \\
&= E(X^2) - 2[E(X)]^2 + E(X)^2 * 1 \\
&= E(X^2) - [E(X)]^2.
\end{aligned}$$

■

Example 17 *Back to the experiment of rolling a pair of dice and summing the faces. The random variable X assigns to each roll its sum. The following table is our previously constructed probability model. Use the shortcut formula to determine standard deviation.*

Sum	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

What is $E(X^2)$?

$$
\begin{aligned}
E(X^2) &= \sum_{x \in X} x * p(x) \\
&= 2^2 * \frac{1}{36} + 3^2 * \frac{2}{36} + 4^2 * \frac{3}{36} + 5^2 * \frac{4}{36} + 6^2 * \frac{5}{36} + 7^2 * \frac{6}{36} \\
&\quad + 8^2 * \frac{5}{36} + 9^2 * \frac{4}{36} + 10^2 * \frac{3}{36} + 11^2 * \frac{2}{36} + 12^2 * \frac{1}{36} \\
&= \frac{4 + 18 + 48 + 100 + 180 + 294 + 320 + 324 + 300 + 242 + 144}{36} \\
&= \frac{329}{6}
\end{aligned}
$$

Next, $\sigma^2 = E(X^2) - [E(X)]^2 = \frac{329}{6} - 7^2 = 5.8333$ and $\sigma = \sqrt{5.8333} = 2.4152$ as advertised above.

Remark 18 *We've shown that $E(aX + b) = a * E(X) + b$. This computing shortcut for standard deviation demonstrates that $E(X^2) \neq [E(X)]^2$.*

1 Exercises

Section 3.4: 28 a, b, c, 29, 31a, b, c, 32, 34, 35, 36, 37, 40