Section 3.6: The Binomial Probability Distribution

The normal distribution is one that occurs frequently in statistics. Others exist as well. One very common discrete distribution is the Binomial Probability Distribution.

Definition 1 *An experiment consisting of repeated trials has a **Binomial Probability Distribution** if and only if*

1. *there are only two outcomes for each trial (success or failure) and the probability p of success is fixed;*
2. *there are a fixed number of trials n;*
3. *the trials are independent.*

Exercise 2 *An experiment consists of flipping a fair coin 10 times and counting the number of tails. Does this experiment have a binomial probability distribution?*

Exercise 3 *A multiple choice test contains 20 questions. Each question has four or five choices for the correct answer. Only one of the choices is correct. With random guessing, does this test have a binomial probability distribution?*

Exercise 4 *A multiple choice test contains 20 questions. Each question has five choices for the correct answer. Only one of the choices is correct. When a student trying their best, does this test have a binomial probability distribution?*

Exercise 5 *A multiple choice test contains 20 questions. Each question has five choices for the correct answer. Only one of the choices is correct. With random guessing, does this test have a binomial probability distribution?*

Exercise 6 *It is known that 30% of all students at the University of Knowhere live off campus. An experiment consists of randomly selecting students until a commuter student is found. Does this experiment have a binomial probability distribution?*

Exercise 7 *An experiment consists of asking 20 students what size water they prefer: small, medium or large. All choices are equally likely. Does this experiment have a binomial probability distribution?*

Exercise 8 *An experiment consists of testing the functionality of five different flash drives from a shipment of 50,000 of which 100 are known to be defective. Does this experiment have a binomial probability distribution? Technically, no since the selections are made without replacement. However the sample size is so small compared to the size of the population that for all practical purposes we can consider the selections to be independent.*

Remark 9 *If the sample size is less than 5% of the population size then it is reasonable to treat selections made without replacement as independent selection.*

What does knowing that an experiment has a binomial probability distribution buy us? Quite a lot actually.

Theorem 10 *If an experiment X has a binomial probability distribution with n trials and probability of success p then*

1. *the probability of exactly k successes is $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$;*
2. *the mean or expected value is $E(X) = \mu = np$;*
3. *and the variance $V(X) = np(1-p)$ and standard deviation $\sigma = \sqrt{np(1-p)}$.*

Proof. Let X have a binomial probability distribution with n trials and probability of success p.

1. Which of the n trials yield a success? We can pick any k of them in an unordered fashion in $\binom{n}{k}$ ways. We must succeed on those k selections which has a likelihood of p^k. Next we must fail on the remaining $n-k$ trials. Doing so has likelihood $(1-p)^{n-k}$. Thus, the probability of exactly k successes is $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$.

2. The moment generating function of X is

$$\begin{aligned}
M_X(t) &= \sum_{x \in X} e^{tx} p(x) \\
&= \sum_{k=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{k=0}^{n} \binom{n}{x} \left(pe^t\right)^x (1-p)^{n-x} \\
&= \left(pe^t + 1 - p\right)^n \text{ by the binomial theorem.}
\end{aligned}$$

The binomial theorem states that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{x} a^x b^{n-x}$ for all integers

n and values of a and b.
So,

$$\begin{aligned} M_X^1(t) &= n(pe^t + 1 - p)^{n-1} pe^t \\ &\quad \text{and} \\ M_X^1(0) &= n(pe^0 + 1 - p)^{n-1} pe^0 \\ &= n * 1^n * p * 1 \\ &= np. \end{aligned}$$

3. Now,

$$\begin{aligned} M_X^2(t) &= n(pe^t + 1 - p)^{n-1} pe^t + n(n-1)(pe^t + 1 - p)^{n-2} pe^t pe^t \\ &\quad \text{and} \\ E(X^2) &= M_X^2(0) \\ &= n(pe^0 + 1 - p)^{n-1} pe^0 + n(n-1)(pe^0 + 1 - p)^{n-2} pe^0 pe^0 \\ &= n * 1^{n-1} * p + n(n-1)(1)^{n-2} * p^2 \\ &= np + n(n-1)p^2. \end{aligned}$$

So,

$$\begin{aligned} V(X) &= E(X^2) - (E(X))^2 \\ &= np + n(n-1)p^2 - (np)^2 \\ &= np + np(n-1)p - (np)(np) \\ &= np(1 + np - p - np) \\ &= np(1-p). \end{aligned}$$

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Remark 11 *Our book uses tables of values found in the appendix to compute binomial probabilities. Feel free to use these or the technology/software of your choice. For example, on the TI-83/84 series of calculators, **binompdf**(n, p, k) computes the probability of exactly k successes. The command **binomcdf**(n, p, k) computes the probability of* 0 *or* 1 *or* 2 *or...or* k *successes.*

Example 12 *An experiment consists of flipping a fair coin 10 times and counting the number of tails. Find the mean and standard deviation for this binomial probability distribution.*

Since $n = 10$ and $p = \frac{1}{2}$, $\mu = 10 * \frac{1}{2} = 5$ and $\sigma = \sqrt{10 * \frac{1}{2} * (1 - \frac{1}{2})} = 1.5811$.

Example 13 *An experiment consists of flipping a fair coin 10 times and counting the number of tails. What is the probability of observing exactly five tails?*

Since $n = 10$, $p = \frac{1}{2}$ and $k = 5$, the probability of observing exactly 5 tails is $p(k = 5) = \binom{10}{5} * (\frac{1}{2})^5 \left(1 - \frac{1}{2}\right)^{(10-5)} = 0.246\,09$

Example 14 *An experiment consists of flipping a fair coin 10 times and counting the number of tails. What is the probability of observing at most three tails?*

Here $n = 10$ and $p = \frac{1}{2}$ but $k = 0, 1, 2$, or 3. These are disjoint cases so we use the addition rule four times.

$p(k \leq 3) = \binom{10}{0} * (\frac{1}{2})^0 \left(1 - \frac{1}{2}\right)^{(10-0)} + \binom{10}{1} * (\frac{1}{2})^1 \left(1 - \frac{1}{2}\right)^{(10-1)} + \binom{10}{2} * (\frac{1}{2})^2 \left(1 - \frac{1}{2}\right)^{(10-2)} + \binom{10}{3} * (\frac{1}{2})^3 \left(1 - \frac{1}{2}\right)^{(10-3)} = 0.171\,88$

Example 15 *An experiment consists of flipping a fair coin 10 times and counting the number of tails. What is the probability of observing at least nine tails?*

Here $n = 10$ and $p = \frac{1}{2}$ but $k = 9$ or 10.

$p(k \geq 9) = \binom{10}{9} * (\frac{1}{2})^9 \left(1 - \frac{1}{2}\right)^{(10-9)} + \binom{10}{10} * (\frac{1}{2})^{10} \left(1 - \frac{1}{2}\right)^{(10-10)} = 1.074\,2 \times 10^{-2}$.

Example 16 *An experiment consists of flipping a fair coin 10 times and counting the number of tails. What is the probability the number of tails falls within two standard deviations of the mean?*

Since $5 \pm 2 * 1.58811$ yields the interval $1.823\,8$ to $8.176\,2$, the probability is $\binom{10}{2} * (\frac{1}{2})^2 \left(1 - \frac{1}{2}\right)^{(10-2)} + \binom{10}{3} * (\frac{1}{2})^3 \left(1 - \frac{1}{2}\right)^{(10-3)} + \binom{10}{4} * (\frac{1}{2})^4 \left(1 - \frac{1}{2}\right)^{(10-4)}$
$+ \binom{10}{5} * (\frac{1}{2})^5 \left(1 - \frac{1}{2}\right)^{(10-5)} + \binom{10}{6} * (\frac{1}{2})^6 \left(1 - \frac{1}{2}\right)^{(10-6)} + \binom{10}{7} * (\frac{1}{2})^7 \left(1 - \frac{1}{2}\right)^{(10-7)}$
$+ \binom{10}{8} * (\frac{1}{2})^8 \left(1 - \frac{1}{2}\right)^{(10-8)} = 0.978\,52.$

Problem 17 *Pandora radio offers a thumb print station which consists solely of songs given a thumbs up by the account user. This gets tricky when married couples share an account on communal devices (Roku, fire stick, etc.). Of all songs given a thumbs up, Joe DeMaio is responsible for 60% while Sylvia DeMaio is responsible for the remaining 40% (no overlap between song choices). Of the next 20 songs played, what is the probability that at least half emanate from Sylvia's songs?*

Problem 18 *In 2017, 71% of all full-time KSU undergraduates received some type of need-based financial aid (https://www.usnews.com/best-colleges/kennesaw-state-university-1577). Twenty students are selected at random.*

1. *Find the mean and standard deviation for the number of students who received need-based financial aid.*

2. *Compute the probability that exactly fourteen of the twenty students received need-based financial aid.*

3. *Compute the probability that at least fifteen of the twenty students received need-based financial aid.*

4. *Suppose only two of the twenty KSU students on the intramural water-polo team received need-based financial aid. What does that suggest about water-polo.*