4.3 Expected Value and Moment Generating Functions.

For a discrete random variable we know that $E(X) = \sum_{x \in X} x * p(x)$. The adjustment for the expected value of a continuous random variable is natural.

Definition 1 *For a continuous random variable X with pdf, $f(x)$, the expected value or mean is* $E(X) = \int_{-\infty}^{\infty} x * f(x)dx$.

Example 2 *Consider the uniform distribution on the interval* $[6, 10]$. *Note that* $f(x) = \frac{1}{4}$ *on the interval* $[6, 10]$. *Determine the expected value.* $E(X) = \int_{-\infty}^{\infty} x * f(x)dx = \int_{6}^{10} x * \frac{1}{4}dx = \frac{x^2}{8}|_6^{10} = \frac{100-36}{8} = 8.0.$

Example 3 *Consider the uniform distribution the interval* $[a, b]$. *Note that* $f(x) = \frac{1}{b-a}$ *on the interval* $[a, b]$. *Determine the expected value.* $E(X) = \int_{-\infty}^{\infty} x * f(x)dx = \int_{a}^{b} x * \frac{1}{b-a}dx = \frac{x^2}{2(b-a)}|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{(b+a)}{2}.$

Example 4 *The pdf of a continuous random variable is given by* $f(x) = \left\{ \begin{array}{ll} \frac{3}{2}(1-x^2) & 0 \le x \le 1 \\ 0 & \textit{otherwise} \end{array} \right\}$. *Find* $E(X)$. $E(X) = \int_{-\infty}^{\infty} x * f(x)dx = \int_{0}^{1} x * \frac{3}{2}(1-x^2)dx = \frac{3}{2}\int_{0}^{1} x - x^3 \ dx = \frac{3}{2}(\frac{x^2}{2} - \frac{x^4}{4}|_0^1) = \frac{3}{2}(\frac{1^2}{2} - \frac{1^4}{4}) = \frac{3}{2} * \frac{1}{4} = \frac{3}{8} = 0.375.$

Definition 5 *The variance of X with pdf $f(x)$ is given by* $V(X) = \int_{-\infty}^{\infty} (x - E(X))^2 * f(x)dx.$

Example 6 *Consider the uniform distribution on the interval* $[6, 10]$. *Note that* $f(x) = \frac{1}{4}$ *on the interval* $[6, 10]$ *and* $E(X) = 8$. *Determine the variance and standard deviation.* $V(X) = \int_{-\infty}^{\infty} (x - E(X))^2 * f(x)dx = \int_{6}^{10} (x-8)^2 \frac{dx}{4} = \frac{1}{4}\int_{6}^{10} (x^2 - 16x + 64)dx = \frac{1}{4}(\frac{x^3}{3} - 8x^2 + 64x|_6^{10}) = \frac{1}{4}((\frac{10^3}{3} - 8 * 10^2 + 64 * 10) - (\frac{6^3}{3} - 8 * 6^2 + 64 * 6)) = \frac{4}{3}$. *Subsequently* $\sigma = \sqrt{\frac{4}{3}} = 1.1547.$

Example 7 *Consider the uniform distribution the interval* $[a, b]$. *Note that* $f(x) = \frac{1}{b-a}$ *on the interval* $[a, b]$ *and* $E(X) = \dfrac{(b+a)}{2}$. *Determine the variance and standard deviation.*

$$
\begin{aligned}
V(X) &= \int_{-\infty}^{\infty} (x - E(X))^2 * f(x)dx \\
&= \int_a^b \left(x - \frac{(b+a)}{2}\right)^2 * \frac{1}{b-a}dx \\
&= \frac{1}{b-a}\int_a^b \left(x - \frac{(b+a)}{2}\right)^2 dx \\
&= \frac{1}{b-a} * \left(\frac{\left(x - \frac{(b+a)}{2}\right)^3}{3}\Big|_a^b\right) \\
&= \frac{1}{b-a} * \frac{1}{3}\left(\left(b - \frac{(b+a)}{2}\right)^3 - \left(\left(a - \frac{(b+a)}{2}\right)^3\right)\right) \\
&= \frac{1}{b-a} * \frac{1}{3}\left(\left(\frac{(b-a)}{2}\right)^3 - \left(\frac{(a-b)}{2}\right)^3\right) \\
&= \frac{1}{b-a} * \frac{1}{3}\left(\left(\frac{(b-a)}{2}\right)^3 + \left(\frac{(b-a)}{2}\right)^3\right) \\
&= \frac{1}{b-a} * \frac{1}{3}\left(2\left(\frac{(b-a)}{2}\right)^3\right) \\
&= \frac{1}{b-a} * \frac{1}{3} * \frac{(b-a)^3}{4} = \\
&= \frac{(b-a)^2}{12}.
\end{aligned}
$$

Thus, $\sigma = \sqrt{\frac{(b-a)^2}{12}} = \frac{b-a}{\sqrt{12}}$.

As before we may wish to compute the expected value of some linear function $h(x)$ based on the pdf $f(x)$. $E(h(X)) = \int_{-\infty}^{\infty} h(x) * f(x)dx$.

Example 8 *The number of tons of copper mined per week depends on the quality of the weather that may impede progress. The quality of the weather is measured by a real number between 0 and 3 with 0 being perfect and 3 being disastrous. We can mine* $30 - x^3$ *tons of copper per week depending on the weather.*

1. Find the value c such that $f(x) = c(30 - x^3)$ is a pdf on $[0, 3]$.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} c * f(x) dx \\ &= c\int_{-0}^{3} (30 - x^3) dx \\ &= c\left(30x - \frac{x^4}{4}\Big|_0^3\right) \\ &= c\left(30 * 3 - \frac{3^4}{4}\right) \\ &= \frac{279}{4}c \end{aligned}$$

So, $c = \dfrac{4}{279}$.

2. What is the expected amount of copper mined during a week?

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x * f(x) dx \\ &= \int_{-0}^{3} x * \frac{4}{279}(30 - x^3) dx \\ &= \frac{4}{279}\int_{-0}^{3} (30x - x^4) dx \\ &= \frac{4}{279}(15x^2 - \frac{x^5}{5}\Big|_0^3) \\ &= \frac{4}{279} * (15 * 3^2 - \frac{3^5}{5}) = \frac{192}{155} \end{aligned}$$

tons of copper.

3. If our profit is based on the spot price of copper (\$3.14 per pound on 10/20/2020) and a fixed operating costs of \$3,500, how much profit do we expect to earn? Our profit function is $h(x) = 1000 * 3.14x - 3500$ per x

tons of copper. So,

$$\begin{aligned}
E(h(X)) &= \int_{-\infty}^{\infty} h(x) * f(x) dx \\
&= \frac{4}{279} \int_{-0}^{3} (1000 * 3.14x - 3500)(30 - x^3) dx \\
&= \frac{4}{279} \int_{-0}^{3} -3140x^4 + 3500x^3 + 94200x - 105\,000 \; dx \\
&= \frac{4}{279} \left(\frac{-3140x^5}{5} + \frac{3500x^4}{4} + \frac{94200x^2}{2} - 105\,000x|_0^3 \right) \\
&= \frac{4}{279} \left(\frac{-3140 * 3^5}{5} + \frac{3500 * 3^4}{4} + \frac{94200 * 3^2}{2} - 105\,000 * 3 \right) \\
&= 389.55.
\end{aligned}$$

Note that $h(\frac{192}{155}) = 1000 * 3.14 * \frac{192}{155} - 3500 = 389.55$

Recall that the moment generating function (mgf) of a discrete random variable X is defined to be

$$\begin{aligned}
M_x(t) &= E(e^{tX}) \\
&= \sum_{x \in X} e^{tx} p(x).
\end{aligned}$$

If X is continuous we cannot list all the values x.

Definition 9 *The moment generating function (mgf) of a continuos random variable X is defined to be*

$$\begin{aligned}
M_x(t) &= E(e^{tX}) \\
&= \int_{-\infty}^{\infty} e^{tx} f(x) \; dx.
\end{aligned}$$

In the discrete case, $M_x(0) = \sum_{x \in X} e^0 p(x) = \sum_{x \in X} p(x) = 1$. Note that in the continuous case, $M_x(0) = \int_{-\infty}^{\infty} e^{0*t} f(x) \; dx = \int_{-\infty}^{\infty} f(x) \; dx = 1$ since $f(x)$ is a pdf.

Example 10 *Find $E(X)$ and $V(X)$ for the distribution with $M_x(t) = \dfrac{2}{(2-t)} = 2 * (2-t)^{-1}$ for $t < 2$. $M_x^1(t) = 2(-1(2-t)^{-2}(-1)) = \dfrac{2}{(2-t)^2}$. Thus, $E(X) =$*

$M_x^1(0) = \frac{1}{2}$. $M_x^2(t) = \dfrac{0 - 2 * 2(2-t)(-1)}{\left((2-t)^2\right)^2} = \dfrac{4(2-t)}{(2-t)^4} = \dfrac{4}{(2-t)^3}$. *Thus,* $E(X^2) = M_x^2(0) = \dfrac{4}{(2-0)^3} = \dfrac{1}{2}$. *Hence,* $V(X) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.