Section 5.2: Jointly Distributed Random Variables

Back to discrete random variables. The pdf of a single discrete random variable X, determines how much probability lands on a single value x. The joint probability distribution of two discrete distributions X and Y has a pdf that determines probability for ordered pairs of points (x, y). Rather than looking at a table for one variable and corresponding probabilities we have a two dimensional table. We've sort of dome something like this before back when we first looked at samples spaces.

Example 1 *Let X represent the outcome of the first die roll and Y represent the outcome of the second die roll. Looking at the sum of the two die faces is a joint probability distribution. Each outcome is equally likely so the pdf is* $f(x, y) = \frac{1}{36}$.

$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(6,1)$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	$(6,3)$
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	$(6,4)$
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	$(6,5)$
$(1,6)$	$(2,6)$	$(3,6)$	$(4,6)$	$(5,6)$	$(6,6)$

Compute each of the following probabilities.

1. $f((2,3)) = \frac{1}{36}$
2. $f(x = 3, y \geq 5) = \sum\limits_{x=3} \sum\limits_{y \geq 5} \frac{1}{36} = \sum\limits_{x=3} 2 * \frac{1}{36} = \sum\limits_{x=3} \frac{2}{36} = 1 * \frac{2}{36} = \frac{1}{18}$
3. $f(x \geq 4, y \geq 5) = \sum\limits_{x \geq 4} \sum\limits_{y \geq 5} \frac{1}{36} = \sum\limits_{x \geq 4} 2 * \frac{1}{36} = \sum\limits_{x \geq 4} \frac{2}{36} = 3 * \frac{2}{36} = \frac{6}{36} = \frac{1}{6}$

Remark 1 *A discrete function $f(x, y)$ is a joint pdf if $f(x, y) \geq 0$ and $\sum\limits_x \sum\limits_y f(x, y) =$ 1.*

Problem 1 *Is $f(x, y) =$*

$X \mid Y$	a	b	c
α	.3	.2	.1
β	.1	.2	.3

a pdf? No since $\sum\limits_x \sum\limits_y f(x, y) =$ 1.2.

Problem 2 *Is* $f(x,y) =$

$X\|Y$	1	2	3
1	.1	.2	.1
2	.1	.2	.3

a pdf? Yes, since $\sum_x \sum_y f(x,y) = 1$ *and* $f(x,y) \geq 0$.

Problem 3 *We know that* $f(x,y) =$

$X\|Y$	1	2	3
1	.1	.2	.1
2	.1	.2	.3

is a pdf.

1. *What is* $f((2,3))$*?* $f((2,3)) = .3$.
2. *What is* $f((x=2, y \geq 2))$*?* $f((x=2, y \geq 2)) = \sum_{x=2} \sum_{y \geq 2} f(x,y) = \sum_{y \geq 2} f(2,y) = f(2,2) + f(2,3) = .2 + .3 = .5$.
3. *What is* $f((x,x))$*?* $f((x,x)) = \sum_x \sum_{y=x} f(x,y) = \sum_x f(x,x) = f(1,1) + f(2,2) = .1 + .2 = .3$.

Definition 2 *The* ***marginal pdf*** *of* $f(x,y)$ *is found by fixing one variable as a constant and summing over all possible values of the other variable. Note that marginal pdfs are just row or column sums for a discrete joint distribution.*

Example 2 *For* $f(x,y) =$

$X\|Y$	1	2	3
1	.1	.2	.1
2	.1	.2	.3

compute

1. the marginal pdf of $x = 1$. $f_X(1) = \sum_{x=1} \sum_y f(x,y) = \sum_y f(1,y) = f(1,1) + f(1,2) + f(1,3) = .1 + .2 + .1 = .4$
2. the marginal pdf of $y = 3$. $f_Y(3) = \sum_x \sum_{y=3} f(x,y) = \sum_x f(x,3) = f(1,3) + f(2,3) = .1 + .3 = .4$

Next we rinse and repeat for continuous random variables. We equate probability to area under the curve for a single valued continuous function $f(x)$. We do the same here with volume in a bounded region for the mutli-value function $f(x,y)$. As before $f(x,y) \geq 0$ and $\int \int f(x,y)\, dx\, dy = 1$.

Example 3 *Is* $f(x,y) = x + y$ *a pdf on* $0 \leq x \leq 1$ *and* $0 \leq x \leq 1$*? The* *function is always non-negative. How about total volume?*

$$
\begin{aligned}
& \int\int f(x,y)\ dx\ dy \\
= & \int_0^1\int_0^1 x+y\ dx\ dy \\
= & \int_0^1 xy+\frac{y^2}{2}\ dx|_0^1 \\
= & \int_0^1 (x+\frac{1^2}{2})-(x*0+\frac{0^2}{2})\ dx \\
= & \int_0^1 (x+\frac{1}{2})\ dx \\
= & \frac{x^2}{2}+\frac{x}{2}|_0^1 \\
= & (\frac{1^2}{2}+\frac{1}{2})-(\frac{0^2}{2}+\frac{0}{2}) = \\
= & 1
\end{aligned}
$$

Example 4 *Find the constant c that makes $f(x,y) = c(x+y)$ a pdf on $0 \leq x \leq 1$ and $0 \leq y \leq 2$.*

$$
\begin{aligned}
1 &= \int\int cf(x,y)\ dx\ dy \\
&= c\int_0^2\int_0^1 x+y\ dx\ dy \\
&= c\int_0^1 xy+\frac{y^2}{2}\ dx\ |_0^2 \\
&= c\int_0^1 xy+\frac{y^2}{2}\ dx\ |_0^2 \\
&= c\int_0^1 (2x+\frac{2^2}{2})-(x*0+\frac{0^2}{2})\ dx \\
&= c\int_0^1 (2x+2)\ dx \\
&= c\left(x^2+2x|_0^1\right) \\
&= c((1^2+2*1)-(0^2+2*0)) \\
&= 3c
\end{aligned}
$$

and $c = \frac{1}{3}$.

Example 5 *For the pdf $f(x,y) = \dfrac{x+y}{3}$ on $0 \leq x \leq 1$ and $0 \leq y \leq 2$, find the $P(x \leq 0.5, y \leq 1)$.*

$\int_0^1 \int_0^{.5} \frac{x+y}{3}\ dx\ dy = \frac{1}{3}\int_0^1 \int_0^{.5} x + y\ dx\ dy = \frac{1}{3}\int_0^{.5} xy + \frac{y^2}{2}\ dx\ |_0^1 = \frac{1}{3}\int_0^{.5} x + \frac{1}{2}$
$dx = \frac{1}{3} * (\frac{x^2}{2} + \frac{x}{2}|_0^{.5}) = \frac{1}{3} * \left(\frac{.5^2}{2} + \frac{.5}{2}\right) = 0.125$

We can also compute marginal pdfs in continuous distributions. We do so by fixing one variable and integrating over the other.

Example 6 *For the pdf* $f(x,y) = \dfrac{x+y}{3}$ *on* $0 \le x \le 1$ *and* $0 \le y \le 2$, *find the marginal pdf of* X, *denoted* $f_X(x)$.
$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\ dy = \int_0^2 \frac{x+y}{3} dy = \frac{1}{3}(xy + \frac{y^2}{2}|_0^2) = \frac{1}{3}(2x + \frac{2^2}{2}) = \frac{2}{3}x + \frac{2}{3}.$

Example 7 *For the pdf* $f(x,y) = \dfrac{x+y}{3}$ *on* $0 \le x \le 1$ *and* $0 \le y \le 2$, *find the marginal pdf of* Y, *denoted* $f_Y(y)$.
$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\ dx = \int_0^1 \frac{x+y}{3} dx = \frac{1}{3}(xy + \frac{x^2}{2}|_0^1) = \frac{1}{3}(y + \frac{1}{2}) = \frac{1}{3}y + \frac{1}{6}.$

Marginal pdfs allow us to extend the definition of independent variables to joint distributions. Recall that for events A and B, A and B are independent if and only if $P(A \cap B) = P(A) * P(B)$. For joint distributions we define random variables X and Y to be independent if and only if $f(x,y) = f_X(x) * f_Y(y)$ for joint random variables.

Example 8 *We know that* $f(x,y) =$

$X\|Y$	1	2	3
1	*.1*	*.2*	*.1*
2	*.1*	*.2*	*.3*

is a pdf. Are X *and* Y *independent variables? First, let's start off with the question is* $f(1,1) = f_X(1) * f_Y(1)$?
$f_X(1) * f_Y(1) = .4 * .2 = 0.08$ *while* $f(1,1) = .1$. *So* $f(1,1) \neq f_X(1) * f_Y(1)$, *and thus* X *and* Y *are not independent. To show that* X *and* Y *are independent we would need to check all six cases of values of* $f(x,y)$.

Example 9 *For the pdf* $f(x,y) = \dfrac{x+y}{3}$ *on* $0 \le x \le 1$ *and* $0 \le y \le 2$, *are* X *and* Y *independent? Note that*

$$\begin{aligned} & f_X(x) * f_Y(y) \\ = & (\frac{2}{3}x + \frac{2}{3}) * (\frac{1}{3}y + \frac{1}{6}) \\ = & \frac{1}{9}x + \frac{2}{9}y + \frac{2}{9}xy + \frac{1}{9} \\ \neq & \frac{x+y}{3}. \end{aligned}$$

and the random variables are not independent. To show that X *and* Y *are independent we would need to symbolically have* $f(x,y) = f_X(x) * f_Y(y)$ *and the bounded regions must form a rectangle whose sides are parallel to the coordinate axes.*

Example 10 *Consider the joint pdf* $f(x,y) = \frac{1}{36}$ *for the faces of two dice* X *and* Y*. Since* $f_X(x) = f_Y(y) = \frac{1}{6}$*,* X *and* Y *are independent variables.*

Example 11 *Consider the joint pdf* $f(x,y) =$

$X\|Y$	1	2
1	*.2*	*.2*
2	*.3*	*.3*

. Are X *and* Y *independent?*

$$\begin{aligned} f(1,1) &= .2;\ f_X(1) * f_Y(1) = .4 * .5 = .2 \\ f(1,2) &= .2;\ \ f_X(1) * f_Y(2) = .4 * .5 = .2 \\ f(2,1) &= .3;\ f_X(2) * f_Y(1) = .6 * .5 = .3 \\ f(2,2) &= .3;\ f_X(2) * f_Y(2) = .6 * .5 = .3 \end{aligned}$$

and yes, X *and* Y *are independent random variables.*

Example 12 *When inspecting a driver at a traffic stop, the patrol officer checks to see if the driver has alcohol on their breath* (A) *and if they have a valid driver's license* (V)*. Assume the results of these checks are independent Bernoulli functions where it is known that 20% of drivers do not have a valid license and 10% of drivers have alcohol on their breath. Construct the joint distribution table. From the given percentages we know that* $f_V(0) = .2$*,* $f_V(1) = .8$*,* $f_A(0) = .9$*, and* $f_A(1) = .1$*. We initially format the table as*

$V\|A$	0	1	*Marginal Totals*
0			.2
1			.8
	.9	.1	

. Since the random variables are independent we can simply multiply to get each individual entry and

$V\|A$	0	1	*Marginal Totals*
0	.18	.02	.2
1	.72	.08	.8
	.9	.1	

.

When considering the joint distribution or independence of multiple variables, these definitions are extended in the natural way to multiple products of marginal pdfs. A binomial distribution is a discrete random variable X. Extending this idea we consider the multinomial distribution as a joint distribution of two or more binomial distributions.

Example 13 *David and Michelle plan to meet between 7:00 and 7:30 at a theatre for an 8:00 movie. Let* D *represent David's arrival time and* M *represent Michelle's arrival time. Both* M *and* D *are uniform distributions on the interval* $[7, 7.5]$*. Also,* D *and* M *are independent events.*

1. What is the joint pdf? Since D is uniform then its pdf $f(x) = \frac{1}{.5} = 2.0$. This is also the distribution of M. The joint pdf is $f(x,y) = 4$ on $7 \le x \le 7.5$ and $7 \le y \le 7..5$ since the random variables are independent.

2. What is the probability that both David and Michelle arrive before 7:12?
$\int_7^{7.2}\int_7^{7.2} 4\ dx\ dy = \int_7^{7.2} 4y\ |_7^{7.2}\ dx = \int_7^{7.2} 4(7.2-7.0)\ dx = \int_7^{7.2} .8\ dx = .8x|_7^{7.2} = .8 * .2 = .16.$

3. What is the probability that both David and Michelle arrive exactly at 7:30?
The probability is 0 since this is one exact value in a continuous distribution.

4. What is the probability that Michelle arrives before 7:06 and David arrives after 7:24?
$\int_{7.4}^{7.5}\int_7^{7.1} 4\ dx\ dy = \int_{7.4}^{7.5} 4y\ |_7^{7.1}\ dx = \int_{7.4}^{7.5} 4(7.1-7.0)\ dx = \int_{7.4}^{7.5} .4\ dx = .4x|_{7.4}^{7.5} = .4 * .1 = 0.04.$

Definition 3 *A multinomial experiment consists of n fixed trials where each trial can result in only one of r possible outcomes and the trials are independent. We denote the likelihood of outcome i to be p_i which is fixed. Note that $\sum_i p_i = 1$. The joint pdf of these independent variables is $p(x_1, x_2, ...x_r) = \frac{n!}{x_1! * x_2! * ... * x_r!} (p_1)^{x_1} * (p_2)^{x_2} * ... * (p_r)^{x_r}$ where $\sum_i x_i = n$.*

Example 14 *A large bag of Hersey's miniatures contains multiple pieces in equal ratios of four types of candies: milk chocolate, dark chocolate, Mr. Goodbar and Krackel. When picking four candies from a bag what is the probability that you select*

1. Four dark chocolates: $\frac{4!}{4! * 0! * 0! * 0!} \left(\frac{1}{4}\right)^4 * \left(\frac{1}{4}\right)^0 * \left(\frac{1}{4}\right)^0 * \left(\frac{1}{4}\right)^0 = 1 * \left(\frac{1}{4}\right)^4 = \frac{1}{256}$;

2. two Mr. Goodbars and two Krackels: $\frac{4!}{2! * 2! * 0! * 0!} * \left(\frac{1}{4}\right)^2 * \left(\frac{1}{4}\right)^2 * \left(\frac{1}{4}\right)^0 * \left(\frac{1}{4}\right)^0 = 6 * \left(\frac{1}{4}\right)^4 = \frac{3}{128}$;

3. one of each candy: $\frac{4!}{1! * 1! * 1! * 1!} * \left(\frac{1}{4}\right)^1 * \left(\frac{1}{4}\right)^1 * \left(\frac{1}{4}\right)^1 * \left(\frac{1}{4}\right)^1 = 24 * \left(\frac{1}{4}\right)^4 = \frac{3}{32}$.

Where does $\frac{n!}{x_1! * x_2! * ... * x_r!}$ come from? With n picks we need to select x_i of each type i. This can be done in $\binom{n}{x_1} * \binom{n}{x_2} * \binom{n}{x_3} * ... * \binom{n}{x_r}$ ways. Plugging and chugging into the formula for binomial coefficients we get

$$
\begin{aligned}
& \binom{n}{x_1} * \binom{n - x_1}{x_2} * \binom{n - x_1 - x_2}{x_3} * ... * \binom{n - x_1 - x_2 - ... - x_{r-1}}{x_r} \\
= \quad & \frac{n!}{x_1!(n - x_1)!} * \frac{(n - x_1)!}{x_2!(n - x_1 - x_2)!} * \frac{(n - x_1 - x_2)!}{x_3!(n - x_1 - x_2 - x_3)!} * ... * \frac{(n - x_1 - x_2 - ... - x_{r-1})!}{x_r!(n - x_1 - x_2 - x_3 ... - x_{r-1} - x_r)!} \\
= \quad & \frac{n!}{x_1! * x_2! * ... * x_r! * 0!} \\
= \quad & \frac{n!}{x_1! * x_2! * ... * x_r!}.
\end{aligned}
$$