Section 5.4: Conditional Distributions

When determining the conditional probability that event A occurs given that event B has occurred, we are very familiar with the formula $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$. Similar forms are used for conditional distributions in joint pdfs.

Definition 1 *For a joint pdf $f(x, y)$ and marginal X pdf $f_X(x)$ and any x where $f_X(x) > 0$, the conditional pdf of Y given X is*

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}.$$

Example 1 *Going back to looking at the sum of the two die faces as a joint probability distribution. Each outcome is equally likely so the pdf is* $f(x,y) = \frac{1}{36}$.

$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(6,1)$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	$(6,3)$
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	$(6,4)$
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	$(6,5)$
$(1,6)$	$(2,6)$	$(3,6)$	$(4,6)$	$(5,6)$	$(6,6)$

What is the probability the sum of the dice is 4? This is straight up probability and is the sum of three cases, $f(1,3) + f(2,2) + f(3,1) = \frac{3}{36} = \frac{1}{12}$.
What is the probability the sum of the dice is 4, given that the first die rolled is 2?
The marginal distribution at $f_X(2) = \frac{6}{36}$. *The probability that the sum is a 4 and the first die is a 2 is* $f(2,2) = \frac{1}{36}$. *Thus,* $f_{Y|X}(y|x) = \dfrac{f(2,2)}{f_X(2)} = \dfrac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$.
What is the probability the sum of the dice is 4, given that the first die rolled is 5? The marginal distribution remains the same at $f_X(2) = \frac{6}{36}$. *However,* $f(5,-1)$ *is not part of the domain of the joint function. Thus,* $f(5,-1) = 0$. *So, Thus,* $f_{Y|X}(y|x) = \dfrac{f(5,-1)}{f_X(2)} = \dfrac{0}{\frac{6}{36}} = 0$.

Example 2 *We know that* $f(x,y) =$

$X\|Y$	1	2	3
1	*.1*	*.2*	*.1*
2	*.1*	*.2*	*.3*

is a pdf. What is the probability that $x = 2$ *given that* $y = 2$? *The marginal distribution of* $f_Y(2) = .2 + .2 = .4$. *The probability that* $x = 2 = y$ *is* $f(2,2) = .2$. *Thus,* $f_{X|Y}(x|y) = \dfrac{f(2,2)}{f_Y(2)} = \frac{.2}{.4} = 0.5$.

Example 3 *We know that* $f(x,y) = x+y$ *is a pdf on* $0 \le x \le 1$ *and* $0 \le y \le 1$. *What is* $P(x < \frac{1}{2} |\ y = \frac{1}{4})$? *The marginal distribution of*

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\ dx \\ &= \int_0^1 x + y\ dx \\ &= \frac{x^2}{2} + yx|_0^1 \\ &= y + \frac{1}{2}. \end{aligned}$$

So, $f_Y(\frac{1}{4}) = \frac{3}{4}$. *Now* $f_{Y|X}(y|x) = \int_{-\infty}^{\infty} \frac{x+y}{\frac{3}{4}} = \frac{4}{3}\int_0^{\frac{1}{2}} x + \frac{1}{4}\ dx = \frac{4}{3}(\frac{x^2}{2} + \frac{x}{4}|_0^{\frac{1}{2}}) = \frac{4}{3}(\frac{(\frac{1}{2})^2}{2} + \frac{\frac{1}{2}}{4}) = \frac{4}{3}(\frac{\frac{1}{4}}{2} + \frac{\frac{1}{2}}{4}) = \frac{4}{3}(\frac{\frac{2}{4}}{4} + \frac{\frac{1}{2}}{4}) = \frac{4}{3}(\frac{1}{4}) = \frac{1}{3}$