Unordered Selections with Repetition

Is order important? Is repetition allowed? Combinatorialists ask these two questions when first faced with a problem. The four possible pairs of answers to these two questions represent four types of problems. The following examples illustrate the four types of problems involved.

A bag of Hershey's Assorted Miniatures contains a supply of four different candies: Hershey's Milk Chocolate, Hershey's Dark Chocolate, Krackel and Mr. Goodbar. Suppose there is exactly one of each candy remaining and there are three children who will choose one candy each. How many different selections exist? The first child has four choices which will leave three choices for the second child and two choices for the last child. This yields $4 * 3 * 2 = 24$ different possible selections. The order of selection is important because each child is distinct and repetition is not allowed with only one piece of each candy available.

Unlike the previous example, suppose there is an ample supply of each candy. The three children still choose one candy each, but it is possible to select the same type of candy as a previous child. How many different selections exist? In this case each child would have all four types of candy from which to select and there would be $4^3 = 64$ different possible selections. Again, the order of the selection is still important because each child is distinct, but now repetition is allowed.

Next, consider the parent of the three children who selects three pieces of candy to distribute at some future time. The order of selection is unimportant because the parent is not designating which piece is for which child. If there is exactly one of each candy remaining then repetition is not allowed. This results in $\binom{4}{3} = 4$ different selections that can be made.

So far, no new counting techniques have been needed. Now, suppose the parent in the previous example selects three pieces of candy to distribute at some future time but there is an ample supply of each type of candy available. How many different unordered selections of three candies can be made? With such a small number of children and types of candy, brute force can be utilized to construct all possible selections.

No repetition	Two milk chocolate	Two dark chocolate	Two Krackel	Two Mr. Goodbar	Three of one type
{M, D, K}	{M, M, D}	{D, D, M}	{K, K, M}	{G, G, M}	{M, M, M}
{M, D, G}	{M, M, K}	{D, D, K}	{K, K, D}	{G, G, D}	{D, D, D}
{M, K, G}	{M, M, G}	{D, D, G}	{K, K, G}	{G, G, K}	{K, K, K}
{D, K, G}					{G, G, G}

Thus, there are twenty possible selections, although, it has been pointed out that the experienced parent would only consider selecting three of one type of candy in order to keep the peace among all the children.

The trick to elegantly count all the different cases is in the representation of a selection of three candies. There are four different types of candies and it is possible to separate these different types by using three bars, |.

M | D | K | G

The selection of three candies is represented by placing three checks, ✓, into the different categories. The selection of two dark chocolates and a Mr. Goodbar is represented by

| ✓✓ | |✓.

The arrangement

✓| |✓|✓

represents the selection of a milk chocolate, a Krackel and a Mr. Goodbar. There are three bars and three checks to arrange into six positions. This can be done in $\binom{6}{3} = 20$ different ways. Hence, there are twenty different unordered selections of three pieces of candy from four types of candy with repetition allowed.

This representation can be easily extended to the general case of selecting *k* objects from *n* types of objects with repetition. With *n* types of objects, $n-1$ bars will be required to separate the object types.

Type 1 | Type 2 | Type 3 | ... | Type $n-1$ | Type *n*

The unordered selection of *k* objects will correspond to the placement of *k* checks among those $n-1$ bars. Adding the number of checks and bars together results in $n+k-1$ positions for all items to occupy. From the total number of positions, select the *k* positions the checks will occupy and Theorem 1.7.1 follows.

Theorem: There are $\binom{n+k-1}{k}$ different ways to choose *k* objects from *n* types of objects as an unordered subset with possible repetition.

The parent in the original problem will have 4 different types of objects (candy) to choose from and will select 3 objects with possible repetition. There will be $\binom{n+k-1}{k} = \binom{4+3-1}{3} = \binom{6}{3} = 20$ different selections.

Continuing with a food example, suppose a local pizza delivery offers a *build-your-own* special. The customer selects any six of the following toppings: mushrooms, onions, green peppers, red peppers, olives, sun dried tomatoes, broccoli, pineapple, pepperoni, sausage, ground beef and ham. An extra helping of any one topping counts as an additional selection. For example, a customer may select three helpings of onions and three helpings of pepperoni as her six toppings. How many different *build-your-own* pizzas can be created? As noted earlier, the order of the toppings is not important and in this situation repeated toppings are allowed. The selection of 6 toppings is made from 12 options with repetition allowed. There are $\binom{12+6-1}{6} = \binom{17}{6} = 12376$ different *build-your-own* pizzas.

Sometimes the selections are restricted. Suppose the parent of the three children decides to select twenty pieces of candy to distribute at a future time. But the parent wants to ensure he has at least three pieces of each type of candy. How many possible selections can be made? First, he selects three pieces of each of the four types of candy. This can be done in one way. Second, 8 pieces of candy must to be selected to obtain a total of twenty pieces. Using 4 types of objects and 8 selections, $\binom{4+8-1}{8} = \binom{11}{8} = 165$ is the number of ways to perform this task. Therefore, there are 165 selections of twenty pieces of candy with at least three pieces of each type of candy.

As can be seen, subtle changes in the wording of a problem result in totally different types of solutions. Use the questions at the beginning of the section to guide you.

Homework

1. A freshman member of a student organization is sent to purchase two dozen donuts for the next club meeting. An ample supply of each of the donut types (chocolate icing, vanilla icing, jelly-filled, creme-filled and plain) is available. How many different ways can the freshman purchase the two dozen donuts?

2. The Jones family is going to fill up their cooler to its twelve can capacity with sodas from a machine. Soda choices are Coke, Diet Coke, Cherry Coke, Caffeine Free Coke, Caffeine Free Diet Coke, Sprite, Ginger Ale, and Surge. How many different ways can the Jones family cooler be filled with drinks?

3. Susan buys an economy pack of fifty pens. The pens are identical except for color. There are ten of each of five different colors.

   i. How many different ways can Susan select four pens of different colors to take to work?

   ii. How many different ways can Susan select ten pens to take to work?

   iii. How many different ways can Susan select ten pens to take to work with at least one of each color?

4. How many different ways can Susan (from problem number 3) select twelve pens from her economy pack to take to work?

5. The parent of the four children selects four pieces of candy to distribute at some future time from a bag that contains five Hershey's Milk Chocolates, three Hershey's Dark Chocolates, five Krackels and three Mr. Goodbars. How many different selections with repetition are possible?

6. A Tim Duncan Topps Finest rookie card is number 101 in a set of 163. Six randomly selected and equally likely cards come in each pack. Of course, with random selection there is the chance that a pack may contain duplicate copies of the same card.

What is the probability that any given pack contains one Duncan rookie card?

What is the probability that any given pack contains at least one Duncan rookie card?

7. Eight letters are randomly selected with possible repetition from the alphabet as a set (perhaps you are playing a popular board game).

What is the probability that the word *dig* can be formed from the chosen letters?

What is the probability that the word *bleed* can be formed from the chosen letters?

What is the probability that the word *level* can be formed from the chosen letters?

8. A game of blackjack in Atlantic City is never played with a single deck of cards. Multiple decks are used to hinder the card counting player. Such a player increases her probability of winning because she keeps track of the cards that have already been dealt.

How many different two card hands of blackjack can be dealt if six decks of cards are used?

What is the probability that blackjack (21) will be dealt if six decks of cards are used?