1 Interlude: Different sizes of $\mathbb{Z}^+$ and $\mathbb{R}$

What do we mean by the size of a set? When sets are finite the answer is easy since the size of a finite set is a number.. The order of a set A, $|A|$, is the number of elements it contains. Let $A = \{1, 2, 3\}$, $B = \{\alpha, \beta, \gamma\}$ and $C = \{\clubsuit, \diamondsuit\}$. Note that $|A| = 3 = |B|$ so A and B have the same size. Since $|C| = 2$, A and C have different sizes.

When sets are infinite, things get trickier since ∞ is a concept rather than a number. We need a different approach that is still consistent with the concept of size of finite sets. A sturdier definition that works with both finite and infinite sets is to say that two sets have the same size if there exists a one-to-one and onto function between the sets. Note that one-to-one and onto functions are invertible. Hence order is a symmetric relation. Using this definition we show that $|A| = |B|$ since

A		B
1	$\rightarrow$	α
2	$\rightarrow$	β
3	$\rightarrow$	γ

is a one-to-one and onto function. In contrast A and C have different sizes since we cannot map all three elements of A to C with a one-to-one function. Conversely, if we attempt to map C to A, no onto function exists. This approach works with sets of infinite size. We define the cardinality of the positive integers as countably infinite. Symbolically, $|\mathbb{Z}^+| = \aleph_0$ (aleph null).

Example 1 *Show that $|\mathbb{Z}| = \aleph_0$. To do so we need to exhibit a one-to-one and onto function between $\mathbb{Z}^+$ and $\mathbb{Z}$. This is easier than it sounds. Here's our function.*

$\mathbb{Z}^+$	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	0	1	-1	2	-2	3	-3	4	...

Note that we could define the rule for this function $f(n) = \left\{ \begin{array}{c} \frac{n}{2} \text{ for even } n \\ -\left\lfloor \frac{n}{2} \right\rfloor \text{ for odd } n \end{array} \right\}$ *but we are not required to do so.*

Example 2 *Show that the size of the even positive integers is $\aleph_0$. All we do is exhibit an appropriate one-to-one and onto function.*

$\mathbb{Z}^+$	1	2	3	4	5	6	7	8	...
$2\mathbb{Z}^+$	2	4	6	8	10	12	14	16	...

Again, we could define the rule for this function $f(n) = 2n$ but we are not required to do so.

Remark 3 *Note that order is a transitive operation. If $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$.*

Example 4 *Show that the size of the even positive integers is the same as the size of the set of all positive integer multiples of 5. We've already shown that $|2\mathbb{Z}^+| = \aleph_0$. We only need to show that $|5\mathbb{Z}^+| = \aleph_0$ and let function composition take care of the rest.*

$\mathbb{Z}^+$	1	2	3	4	5	6	7	8	...
$5\mathbb{Z}^+$	5	10	15	20	25	30	35	40	...

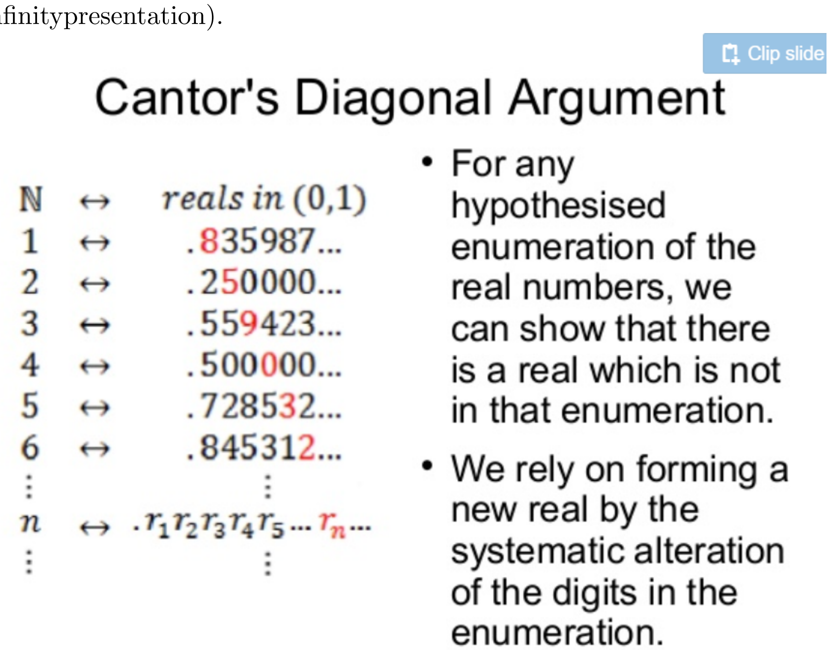
or $g(n) = 5n$. *The function that maps* $2\mathbb{Z}^+ \rightarrow 5\mathbb{Z}^+$ *is* $(g \circ f^{-1})(n) = \frac{5n}{2}$.

Exercise 5 *Let's show that* $|\mathbb{Q}^+| = \aleph_0$.

Now for the fun part! We need to show that the interval of real numbers $(0, 1)$ does not have size $\aleph_0$. The proof is by contradiction. Assume that there is some one-to-one and onto function between $\mathbb{Z}^+$ and $(0, 1)$.

$\mathbb{Z}^+$	1	2	3	4	5	6	7	8	...
$(0, 1)$	?	?	?	?	?	?	?	?	...

If I can point to a number in $(0, 1)$ that is not included in this ordered list then I know that no such one-to-one and onto function exists. The method to do so is known as Cantor's diagonal argument (https://www.slideshare.net/mattspaul/matthew-infinitypresentation).



We will now point to a real number r in $(0, 1)$ that is not in the alleged one-to-one and onto mapping. Let $r = 0.d_1d_2d_3...d_i...$, and thus $r \in (0, 1)$. What is d_i? We let $d_i = 0$ unless the i^{th} digit of the real number mapped to integer i by the assumed one-to-one and onto mapping is 0. If that is the case then $d_i = 1$. For the alleged mapping above, $r = 0.000100....$ Since r always differs by at least one digit from every real in the listing then r is not in the alleged one-to-one and onto function between $\mathbb{Z}^+$ and $(0, 1)$. Thus, the mapping is not onto and the interval of real numbers $(0, 1)$ does not have size $\aleph_0$. We say that the size of $(0, 1)$ is c (for continuum).

2 Exercises

1. Show that the size of the set of all positive integer multiples of 7 has cardinality $\aleph_0$.
2. Show that the size of the set of all positive integer multiples of k has cardinality $\aleph_0$ for any $k \in \mathbb{Z}^+$.
3. Show that the size of the even integers is the same as the size of the set of all positive integer multiples of 5.
4. Let $A = \{1, 2\}$. Show that $|A| \neq |A \times A|$.
5. Show that $|\mathbb{Z}^+| = |\mathbb{Z}^+ \times \mathbb{Z}^+|$. Hint! Use a technique of this section.
6. Give an example of sets A and B such that A is a proper subset of B but $|A| = |B|$.