King's Total Domination Number on the Square of Side n

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Abstract

A set $S \subseteq V$ is a dominating set of a graph $G = (V, E)$ if each vertex in V is either in S or is adjacent to a vertex in S. A vertex is said to dominate itself and all its neighbors. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. In terms of a chess board problem, let X_n be the graph for chess piece X on the square of side n. Thus, $\gamma(X_n)$ is the domination number for chess piece X on the square of side n. In 1964, Yaglom and Yaglom established that $\gamma(K_n) = \left\lfloor \frac{n+2}{3} \right\rfloor^2$. This extends to $\gamma(K_{m,n}) = \left\lfloor \frac{m+2}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor$ for the rectangular board. A set $S \subseteq V$ is a total dominating set of a graph $G = (V, E)$ if each vertex in V is adjacent to a vertex in S. A vertex is said to dominate its neighbors but not itself. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. In 1995, Garnick and Nieuwejaar conducted an analysis of the total domination numbers for the king's graph on the $m \times n$ board. In this paper we note an error in one portion of their analysis and provide a correct general upperbound for $\gamma_t(K_{m,n})$. Furthermore, we state improved upperbounds for $\gamma_t(K_n)$.

1 Introduction

Puzzles on the chessboard have long been studied by mathematicians. The survey papers Combinatorial Problems on Chessboards [4] and Combinatorial Problems on Chessboards II [9] provide excellent introductions to the various types of problems. The history of domination problems on a chessboard can be traced back to Max Bezzel in the 1850's where determining the minimum number of queens needed to occupy or attack every square on the 8×8 board was studied [1]. Naturally, we do not restrict ourselves only to the standard 8×8 chessboard. Generalizations are quickly made to the square board, the rectangular board, etc. In this paper we consider the total domination number on the square and rectangular board by kings.

A set $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if each vertex in V is either in S or is adjacent to a vertex in S. A vertex is said to dominate itself and all its neighbors. A *minimal dominating set* S is a dominating set of G that contains no proper subset that is also a dominating set of G. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S of a graph G such that $|S| = \gamma(G)$ is conveniently called a $\gamma(G)$-set. A set $S \subseteq V$ is a *total dominating set* of a graph $G = (V, E)$ if each vertex in V is adjacent to a vertex in S. For total domination a vertex only dominates its neighbors and no longer dominates itself. A *minimal total dominating* set S is a total dominating set of G that contains no proper subset that is also a total dominating set of G. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. A total dominating set S of a graph G such that $|S| = \gamma_t(G)$ is conveniently called a $\gamma_t(G)$-set. Two bounds on $\gamma_t(G)$ are immediate.

$$\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G) \tag{1}$$

In terms of a chessboard problem, let X_n be the graph for chess piece X on the square of side n. Thus, $\gamma(X_n)$ is the domination number for chess piece X on the square of side n. For example, $\gamma(Q_8) = 5$ indicates that the standard 8×8 board can be dominated with five queens but not four and $\gamma_t(K_5) = 5$ indicates that five kings can form a total dominating set of the 5×5 board but four kings cannot. It is well known that $\gamma(K_n) = \left\lfloor \frac{n+2}{3} \right\rfloor^2$, due to Yaglom and Yaglom [11]. For the rectangular $m \times n$ chessboard this extends to $\gamma(K_{m,n}) = \left\lfloor \frac{m+2}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor$.

2 King's Total Domination on the $m \times n$ Chessboard

Garnick and Nieuwejaar consider total domination by kings in [6] and determine exact values for small n. It is interesting to note that both extremes of Equation 1 on $\gamma_t(G)$ are realized. We see that $\gamma(K_4) = \gamma_t(K_4)$ and $\gamma(K_7) = \gamma_t(K_7)$ while $\gamma_t(K_6) = 2\gamma(K_6)$.

n	2	3	4	5	6	7	8	9	10	11	12
$\gamma_t(K_n)$	2	2	4	5	8	9	12	15	18	21	24

Furthermore, Garnick and Nieuwejaar provide upper bounds for the next few small n.

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$\gamma_t(K_n) \leq$	29	33	38	43	48	54	60	68	72	80	87	95	102

3 A Slightly Flawed General Upper Bound

Garnick and Nieuwejaar give the general upper bound $\gamma_t(K_{m,n}) \leq \frac{mn+2n+89}{7}$ for the rectangular $m \times n$ board. For the square board this becomes $\gamma_t(K_n) \leq \frac{n^2+2n+89}{7}$. Unfortunately there is a mistake in the Garnick and Nieuwejaar upper bound for the general rectangular $m \times n$ board. The strategy they use to obtain this bound is correct and complete but one case is omitted in the final summation of the required number of kings. However, it is not until n grows sufficiently large that this discrepancy is noticed.

First some strategy and definitions of common tiles that appear as we construct total dominating sets on the rectangular chessboard using the Garnick and Nieuwejaar method and an example to show that the Garnick and Nieuwejaar upper bound is too small. For total domination, two kings must be adjacent when placed on the chess board. The maximum number of squares dominated by two adjacent kings is 14 as shown by tile 1 in Figure 1. Two adjacent kings placed on the same row (or column) dominate only 12 squares. In the long run tiling a board with the copies of tile 1 will be more efficient than placing adjacent kings on the same row (or column). Such a tiling will leave gaps of uncovered squares along the exterior sides of the board. These uncovered gaps can always be covered (though sometimes with overlap) with copies of tiles 2 and 3 as shown in Figure 1.

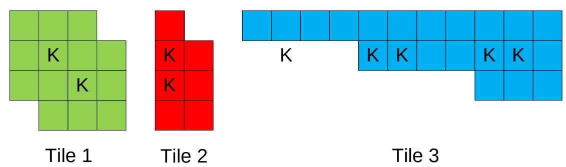

Figure 1: Tiles for Total Domination by Kings

Figure 2 demonstrates the $n = 60$ square board tiling by Garnick and Nieuwejaar's strategy. It will also illustrate the flaw in their upper-bound. For $m = n = 60$, the Garnick and Nieuwejaar bound is $\frac{60^2+2*60+89}{7} = \frac{3809}{7} = 544.14$ kings. Starting in the upper left hand corner, tile 1 is used to cover as much of the board as possible. We need 474 kings to cover the Garnick and Nieuwejaar tiling of 237 copies of tile 1. For tile 2 on the left and right hand sides of the board, $8 * 2 * 2 = 32$ kings are required. This leaves us with $544.14 - (474 + 32) = 38.14$ kings left to dominate the remaining 8 copies of tile 3. However, we need 40 kings for the 8 copies of tile 3 and we still have yet to cover the bottom left hand corner and the upper right hand corner. The Garnick and Nieuwejaar bound is incorrect at least for $n = 60$.

Make no mistake about it; the general analysis by Garnick and Nieuwejaar is correct. The Garnick and Nieuwejaar bound uses $\frac{mn-a}{7} + b$ kings for total domination of the $m \times n$ chessboard where a is the number of uncovered squares remaining after the initial covering using tile 1 and b is the number of kings required to cover those a remaining squares. The problem with the Garnick and Nieuwejaar bound appears to be one of simple addition.

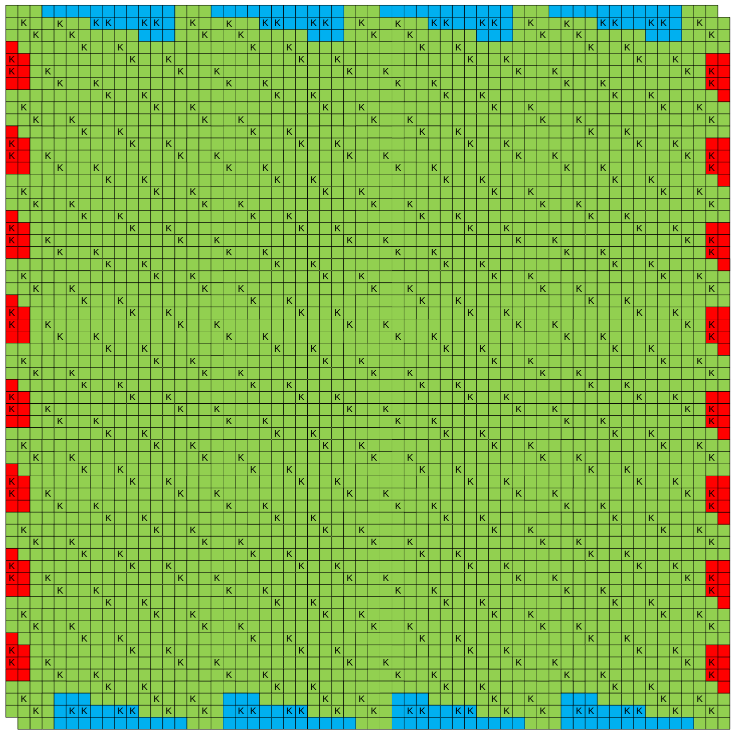

Figure 2: Tiling the Board of Side $n = 60$

3.1 A Corrected General Upper Bound

For even n, there are at least $\left\lfloor \frac{m}{7} \right\rfloor$ uncovered copies of tile 2 running along each of the left and right hand sides of the board. Furthermore, there will be at least $\left\lfloor \frac{n}{14} \right\rfloor$ uncovered copies of tile 3 running along each of the top and bottom of the board. The number of uncovered squares by the initial covering with tile 1 is $a \geq 42 \left\lfloor \frac{n}{14} \right\rfloor + 14 \left\lfloor \frac{m}{7} \right\rfloor$. Each tile 2 can be covered with two kings and each tile 3 can be covered with 5 kings. The number of kings needed to cover the uncovered squares by the initial covering with tile 1 of the board is at most $b \leq 10 \left\lceil \frac{n}{14} \right\rceil + 4 \left\lceil \frac{m}{7} \right\rceil$.[1] For even n,

$$\begin{aligned} \frac{mn-a}{7} + b &= \frac{mn - \left(42\left\lfloor \frac{n}{14} \right\rfloor + 14\left\lfloor \frac{m}{7} \right\rfloor\right)}{7} + 10\left\lceil \frac{n}{14} \right\rceil + 4\left\lceil \frac{m}{7} \right\rceil \\ &= \frac{mn}{7} - 6\left\lfloor \frac{n}{14} \right\rfloor - 2\left\lfloor \frac{m}{7} \right\rfloor + 10\left\lceil \frac{n}{14} \right\rceil + 4\left\lceil \frac{m}{7} \right\rceil \\ &\leq \frac{mn}{7} - 6\left(\frac{n}{14} - 1\right) - 2\left(\frac{m}{7} - 1\right) + 10\left(\frac{n}{14} + 1\right) + 4\left(\frac{m}{7} + 1\right) \\ &= \frac{mn}{7} + 4\frac{n}{14} + 2\frac{m}{7} + 22 = \frac{mn+2n+2m+154}{7}. \end{aligned}$$

For small values of n this is not an impressive bound when compared to the very simplistic $\gamma_t(K_n) \leq 2\gamma(K_n)$. This is due to the 22 extra kings needed when approximating the bound by removing the floor and ceiling functions.[2] For example,

[1] Here is the appearance of the addition error by Garnick and Nieuwejaar. They compute $b \leq 10\left\lceil \frac{n}{14} \right\rceil + 2\left\lceil \frac{m}{7} \right\rceil$ and neglect to add in the $2\left\lceil \frac{m}{7} \right\rceil$ copies of tile 2 along the left-hand side of the board.

[2] Note that $\frac{x}{y} - 1 \leq \left\lfloor \frac{x}{y} \right\rfloor \leq \frac{x}{y} \leq \left\lceil \frac{x}{y} \right\rceil \leq \frac{x}{y} + 1$.

$2\gamma\left(K_{14,18}\right) = 60$ and our upper bound is $\frac{14*18+2*14+2*18+154}{7} = 67.143$. For large n the significance of a small number of extra kings is vastly diminished. Such is the case where $2\gamma\left(K_{130,170}\right) = 4928$ while our bound is 3302.

For odd n, the analysis is quite similar. Once again there are at least $\left\lfloor \frac{m}{7} \right\rfloor$ copies of tile 2 running along the left hand side of the board and at least $\left\lfloor \frac{n}{14} \right\rfloor$ copies of tile 3 running along each of the top and bottom of the board. However on the right hand side of the board there are at least $\left\lfloor \frac{m}{7} \right\rfloor$ copies of an extended tile 2 (tile 11) as shown in Figure 3 of 11 squares plus least $\left\lfloor \frac{m}{7} \right\rfloor$ copies of a three square strip. Now $a \geq 42\left\lfloor \frac{n}{14} \right\rfloor + 21\left\lfloor \frac{m}{7} \right\rfloor$. This extended tile 2 can still be covered with two kings and the new strip of three squares can be covered with a single king (in conjunction with a tile 1). So, $b \leq 10\left\lceil \frac{n}{14} \right\rceil + 5\left\lceil \frac{m}{7} \right\rceil$.[3] For odd n, $\frac{mn-a}{7} + b = \frac{n^2-\left(42\left\lfloor \frac{n}{14} \right\rfloor + 21\left\lfloor \frac{m}{7} \right\rfloor\right)}{7} + 10\left\lceil \frac{n}{14} \right\rceil + 5\left\lceil \frac{m}{7} \right\rceil = \frac{mn}{7} - 6\left\lfloor \frac{n}{14} \right\rfloor - 3\left\lfloor \frac{m}{7} \right\rfloor + 10\left\lceil \frac{n}{14} \right\rceil + 5\left\lceil \frac{m}{7} \right\rceil$
$\leq \frac{mn}{7} - 6\left(\frac{n}{14} - 1\right) - 3\left(\frac{m}{7} - 1\right) + 10\left(\frac{n}{14} + 1\right) + 5\left(\frac{m}{7} + 1\right)$

$= \frac{mn}{7} + 4\frac{n}{14} + 2\frac{m}{7} + 24 = \frac{mn+2n+2m+168}{7}$.

$$\text{In summary, } \gamma_t\left(K_{m,n}\right) \leq \left\{ \begin{array}{ll} \frac{mn+2n+2m+154}{7} & \text{for even } n \\ \frac{mn+2n+2m+168}{7} & \text{for odd } n \end{array} \right\}.$$

4 King's Total Domination on the Square Chessboard of Even Side n

At this point we wish to consider a more detailed analysis for the square board of side n. We can provide a tighter bound for $m = n$ using modified versions of tiles 2 and 3 after the initial tiling of the board with tile 1. The modified versions of tiles 2 and 3 appear in Figure 3. The period of the cycle introduced with the Garnick and Nieuwejaar tiling strategy has length 14. We have seven cases each for even and odd n. We first present the even cases in their natural order for n mod 14. However, $n \equiv 0, 4 \bmod 14$ are the most straightforward cases and upon an introductory analysis should be considered first. We continue to use the Garnick and Nieuwejaar tiling strategy and determine at most $\frac{n^2-a}{7} + b$ kings are needed for total domination of the square board of side n where a is the number of uncovered squares remaining after the initial covering using tile 1 and b is the number of kings required to cover those a remaining squares. Tiles 2 and 3 are utilized for each board and the miscellaneous tiles of Figure 3 will be used as necessary.

[3]Once again is the appearance of the addition error by Garnick and Nieuwejaar. They compute $b \leq 10\left\lceil \frac{n}{14} \right\rceil + 3\left\lceil \frac{m}{7} \right\rceil$ and neglect to add in the $2\left\lceil \frac{m}{7} \right\rceil$ copies of tile 2 along the left-hand side of the board.

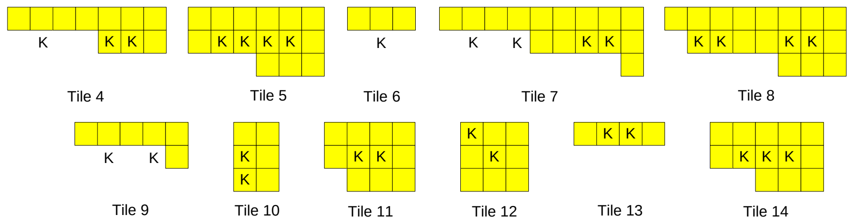

Figure 3: Additional Tiles for Total Domination by Kings

In all cases, each tile 2 removes 7 squares and each tile 3 removes 21 additional squares in our computation for a. In all cases, tile 2 requires two additional kings to cover and tile 3 requires 5 additional kings to cover for b. Both tiles play a significant role for each incongruent value of n mod 14.

4.1 $n \equiv 0 \bmod 14$

For $n \equiv 0 \bmod 14$, $n = 14k$ and $k = \frac{n}{14}$. There are exactly $2k$ copies of tile 2 running down the left side of the board and $2k$ copies of tile 2 running down the right side of the board. Furthermore there are exactly k copies of tile 3 on each the top and bottom of the board. Thus, $a = 28k + 42k = 70k$. Only tiles 2 and 3 are needed and $b = 8k + 10k = 18k$. Thus, at most $\frac{n^2-a}{7} + b = \frac{n^2-70\left(\frac{n}{14}\right)}{7} + 18\left(\frac{n}{14}\right) = \frac{n^2+4n}{7}$ kings are needed. Figure 4 illustrates this tiling for $n = 14$.

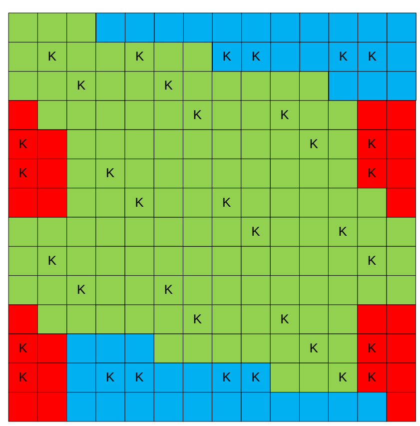

Figure 4: Tiling the Board of Side $n = 14$

4.2 $n \equiv 2 \bmod 14$

For $n \equiv 2 \bmod 14$, $n = 14k + 2$ and $k = \frac{n-2}{14}$. There are exactly $2k$ copies of tile 2 running down the left side of the board and $2k$ copies of tile 2 running down the right side of the board. There are k copies of tile 3 on the top of the board. The bottom tier of the board is more complex. There are $k-1$ copies of tile 3. On the lower left we have a copy of tile 4 with 10 squares and on the lower right there exists a copy of tile 5 with 15 squares. Thus, $a = 28k + 21(2k-1) + 10 + 15 = 70k + 4$. Tile 4 can be covered with 3 kings and tile 5 requires 4 kings. Hence, $b = 8k + 5(2k-1) + 3 + 4 = 18k + 2$. This shows at most $\frac{n^2-a}{7} + b = \frac{n^2-(70(\frac{n-2}{14})+4)}{7} + 18\left(\frac{n-2}{14}\right) + 2 = \frac{n^2+4n+2}{7}$ kings are needed.

4.3 $n \equiv 4 \bmod 14$

For $n \equiv 4 \bmod 14$, $n = 14k+4$ and $k = \frac{n-4}{14}$. There are exactly $2k$ copies of tile 2 running down the left side of the board and $2k$ copies of tile 2 running down the right side of the board. Also the very bottom left and upper right squares of the board are uncovered. Furthermore there are exactly k copies of tile 3 on each the top and bottom of the board. Thus, $a = 28k + 42k + 2 = 70k + 2$ squares. The stray bottom left and upper right corners require a single king each. We need an additional $b = 8k + 10k + 2 = 18k + 2$ kings to cover the rest of the board. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(70\left(\frac{n-4}{14}\right)+2\right)}{7} + 18\left(\frac{n-4}{14}\right) + 2 = \frac{n^2+4n-4}{7}$ kings are required. It is interesting to note that for $n \equiv 4 \bmod 14$, this tiling displays $180°$ symmetry.

4.4 $n \equiv 6 \bmod 14$

For $n \equiv 6 \bmod 14$, $n = 14k+6$ and $k = \frac{n-6}{14}$. There are exactly $2k$ copies of tile 2 running down the left side of the board and $2k+1$ copies of tile 2 running down the right side of the board. On the top of the board there are k copies of tile 3 and on the bottom there are $k-1$ copies of tile 3. In the bottom row on the left corner there is a single copy of tile 6 and on the right corner a single copy of tile 7. Finally there is a single copy of tile 8 in the upper right corner. Hence, $a = 7(4k+1) + 21(2k-1) + 15 + 18 + 3 = 70k+22$. Tiles 6 and 7 each require 4 additional kings and tile 8 requires one additional king. Thus, $b = 2(4k+1) + 5(2k-1) + 4 + 4 + 1 = 18k + 6$. At most $\frac{n^2-a}{7} + b = \frac{n^2-(70(\frac{n-6}{14})+22)}{7} + 18\left(\frac{n-6}{14}\right) + 6 = \frac{n^2+4n-4}{7}$ kings are needed.

4.5 $n \equiv 8 \bmod 14$

For $n \equiv 8 \bmod 14$, $n = 14k+8$ and $k = \frac{n-8}{14}$. There are exactly $2k+1$ copies of tile 2 running down the left side of the board and $2k$ copies of tile 2 running down the right side of the board. There are k copies of tile 3 on both the top and bottom of the board. On the top right corner there is a copy of tile 9. On the bottom left corner there is a tile 6 and on the bottom right there is a copy of tile 10. Hence $a = 7(4k+1) + 42k + 6 + 3 + 6 = 70k + 22$. Tile 9 requires 2 additional kings. Tile 6 requires one additional king and tile 10 requires 2 additional kings. Thus, $b = 2(4k+1) + 5(2k) + 2 + 1 + 2 = 18k+7$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(70\left(\frac{n-8}{14}\right)+22\right)}{7} + 18\left(\frac{n-8}{14}\right) + 7 = \frac{n^2+4n-5}{7}$ kings are needed.

4.6 $n \equiv 10 \bmod 14$

For $n \equiv 10 \bmod 14$, $n = 14k + 10$ and $k = \frac{n-10}{14}$. There are exactly $2k + 1$ copies of tile 2 running down both the left side and the right side of the board. Also, there are k copies of tile 3 on both the top and bottom of the board. In the upper right hand corner there is a copy of tile 4. In the bottom right hand corner there is a copy of tile 3 with the right-most square removed. Thus, $a = 14\,(2k + 1) + 42k + 10 + 20 = 70k + 44$. Tile 4 can be covered with 3 kings and the almost tile 3 in the bottom right corner requires 5 kings. Thus, $b = 2\,(4k + 2) + 10k + 3 + 5 = 18k + 12$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(70\left(\frac{n-10}{14}\right)+44\right)}{7} + 18\left(\frac{n-10}{14}\right) + 12 = \frac{n^2+4n}{7}$ kings are required.

4.7 $n \equiv 12 \bmod 14$

For $n \equiv 12 \bmod 14$, $n = 14k + 12$ and $k = \left(\frac{n-12}{14}\right)$. There are exactly $2k + 1$ copies of tile 2 running down both the left side and the right side of the board. Also, there are k copies of tile 3 on both the top and bottom of the board. In the upper right hand corner we have a copy of tile 7. On the bottom row to the left is a copy of tile 9 and to the right a copy of tile 14. Hence, $a = 14\,(2k + 1) + 42k + 15 + 6 + 11 = 70k+46$. Tile 7 requires 4 kings, tile 9 requires 2 kings and tile 11 requires 2 kings. So $b = 2\,(4k + 2)+10k+4+2+2 = 18k+12$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(70\left(\frac{n-12}{14}\right)+46\right)}{7} + 18\left(\frac{n-12}{14}\right) + 12 = \frac{n^2+4n-10}{7}$ kings are needed.

5 King's Total Domination on the Square Chessboard of Odd Side n

The odd cases of $n \bmod 14$ are very similar to the even cases. Covering the square chessboard of odd side n with tile 1 will leave many uncovered tile 3's along the top and bottom of the board. This tilling will also leave many uncovered tile 2's on the left side of the board. However the right side of the board leaves many modified versions of tile 2 uncovered. This modified version of tile 2 is just a rotated copy of tile 11 with 11 squares and like tile 2, requires only two additional kings. Furthermore, tile 6 appears on the right-hand side of all boards.

5.1 $n \equiv 1 \bmod 14$

For $n \equiv 1 \bmod 14$, $n = 14k + 1$ and $k = \left(\frac{n-1}{14}\right)$. There are exactly $2k$ copies of tile 2 on the left-hand side. On the right-hand side there are exactly $2k$ copies of tile 11 and $2k$ copies of tile 6. On the top of the board there exist k copies of tile 3. On the bottom of the board there are $k - 1$ copies of tile 3. In the left-hand bottom corner there is a copy of tile 6. Finally in the right hand bottom corner there is a shortened copy of tile 3 with 19 squares. Hence $a = 14k + 22k + 6k + 21(2k - 1) + 3 + 19 = 84k + 1$. Thus, $b = 8k + 10k + 2k + 1 = 20k + 1$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(84\left(\frac{n-1}{14}\right)+1\right)}{7} + 20\left(\frac{n-1}{14}\right) + 1 = \frac{n^2+4n+2}{7}$ kings are needed. Figure 5 illustrates this tiling for $n = 15$.

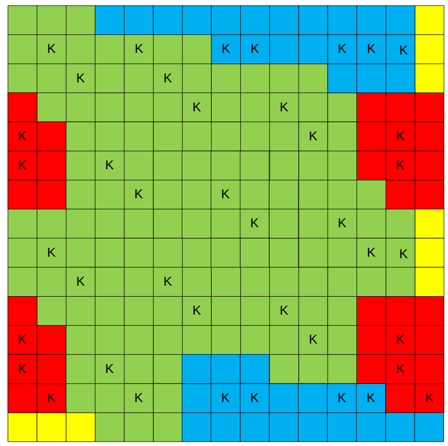

Figure 5: Tiling the Board of Side $n = 15$

5.2 $n \equiv 3 \bmod 14$

For $n \equiv 3 \bmod 14$, $n = 14k + 3$ and $k = \left(\frac{n-3}{14}\right)$. There are exactly $2k$ copies of tile 2 on the left-hand side. On the right-hand side there are exactly $2k$ copies of tile 11 and $2k$ copies of tile 6. On the top of the board there exist k copies of tile 3. On the bottom of the board there are k copies of tile 3. Finally in the right hand bottom corner there is a new tile 12 with 9 squares. Hence $a = 14k + 22k + 6k + 42k + 9 = 84k + 9$. Two kings are needed for the new tile 12. Thus, $b = 8k + 10k + 2k + 2 = 20k + 2$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(84\left(\frac{n-3}{14}\right)+9\right)}{7} + 20\left(\frac{n-3}{14}\right) + 2 = \frac{n^2+4n-7}{7}$ kings are needed.

5.3 $n \equiv 5 \bmod 14$

For $n \equiv 5 \bmod 14$, $n = 14k + 5$ and $k = \left(\frac{n-5}{14}\right)$. There are exactly $2k$ copies of tile 2 on the left-hand side. On the right-hand side there are exactly $2k$ copies of tile 11 and $2k$ copies of tile 6. On the top of the board there exist k copies of tile 3. On the bottom of the board there are k copies of tile 3. In the upper right-hand corner a shortened tile 9 of 5 squares exists. In the bottom left-hand corner a tile 9 exists. Thus, $a = 14k + 22k + 6k + 42k + 6 + 5 = 84k + 11$. Two kings are needed for tile 9 and its shortened version. Hence, $b = 8k + 10k + 2k + 2 + 2 = 20k + 4$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(84\left(\frac{n-5}{14}\right)+11\right)}{7} + 20\left(\frac{n-5}{14}\right) + 4 = \frac{n^2+4n-3}{7}$ kings are needed.

5.4 $n \equiv 7 \bmod 14$

For $n \equiv 7 \bmod 14$, $n = 14k + 7$ and $k = \left(\frac{n-7}{14}\right)$. There are exactly $2k + 1$ copies of tile 2 on the left-hand side. On the right-hand side there are exactly $2k + 1$

copies of tile 11 and $2k$ copies of tile 6. On the top of the board there exist k copies of tile 3. On the bottom of the board there are k copies of tile 3. In the top right-hand corner a new tile 13 of 4 squares exists. In the bottom right-hand corner a new tile 14 of 13 squares exists. Hence, $a = 7\,(2k+1) + 11\,(2k+1) + 6k + 42k + 4 + 13 = 84k + 35$. Two kings are needed for tile 13 and four kings are needed for tile 14. Hence, $b = 2\,(2k+1) + 2\,(2k+1) + 2k + 10k + 2 + 3 = 20k + 9$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(84\left(\frac{n-7}{14}\right)+35\right)}{7} + 20\left(\frac{n-7}{14}\right) + 9 = \frac{n^2+4n}{7}$ kings are needed.

5.5 $n \equiv 9 \bmod 14$

For $n \equiv 9 \bmod 14$, $n = 14k + 9$ and $k = \left(\frac{n-9}{14}\right)$. There are exactly $2k+1$ copies of tile 2 on the left-hand side. On the right-hand side there are exactly $2k+1$ copies of tile 11 and $2k+1$ copies of tile 6. On the top of the board there exist k copies of tile 3. On the bottom of the board there are k copies of tile 3. In the top right-hand corner a shortened tile 4 of 8 squares exists. In the bottom right-hand corner a full tile 4 of 10 squares exists. Hence, $a = 7\,(2k+1) + 11\,(2k+1) + 3(2k+1) + 42k + 8 + 10 = 84k + 39$. Three kings are needed for each of tile four and its shortened form. Thus, $b = 2\,(2k+1) + 2\,(2k+1) + 2k + 1 + 10k + 3 + 3 = 20k + 11$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(84\left(\frac{n-9}{14}\right)+39\right)}{7} + 20\left(\frac{n-9}{14}\right) + 11 = \frac{n^2+4n+2}{7}$ kings are needed.

5.6 $n \equiv 11 \bmod 14$

For $n \equiv 11 \bmod 14$, $n = 14k + 11$ and $k = \left(\frac{n-11}{14}\right)$. There are exactly $2k+1$ copies of tile 2 on the left-hand side. On the right-hand side there are exactly $2k+1$ copies of tile 11 and $2k+1$ copies of tile 6. On the top of the board there exist k copies of tile 3. On the bottom of the board there are k copies of tile 3. In the top right-hand corner a shortened tile 7 of 12 squares exists. In the bottom right-hand corner a short tile 8 of 17 squares exists. Finally a lone square exists in the bottom left-hand corner. Hence, $a = 7\,(2k+1) + 11\,(2k+1) + 3(2k+1) + 42k + 12 + 17 + 1 = 84k + 51$. Four kings are needed for each short tile 7 and 8. One final king for the lower left-hand corner. Thus, $b = 2\,(2k+1) + 2\,(2k+1) + 2k + 1 + 10k + 4 + 4 + 1 = 20k + 14$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(84\left(\frac{n-11}{14}\right)+51\right)}{7} + 20\left(\frac{n-11}{14}\right) + 14 = \frac{n^2+4n+3}{7}$ kings are needed.

5.7 $n \equiv 13 \bmod 14$

For $n \equiv 13 \bmod 14$, $n = 14k + 13$ and $k = \left(\frac{n-13}{14}\right)$. There are exactly $2k+1$ copies of tile 2 on the left-hand side. On the right-hand side there are exactly $2k+1$ copies of tile 11 and $2k+2$ copies of tile 6. On the top of the board there exist k copies of tile 3. On the bottom of the board there are k copies of tile 3. A short tile 3 of 18 squares appears in the upper right hand corner. Tile 7 with 15 squares appears in the right hand corner. Hence $a = 7\,(2k+1) + 11\,(2k+1) + 3(2k+2) + 42k + 18 + 15 = 84k + 57$. Five more kings for the short tile 3 and four kings for tile 7. Thus, $b = 2\,(2k+1) + 2\,(2k+1) + 2k + 2 + 10k + 5 + 4 = 20k + 15$. At most $\frac{n^2-a}{7} + b = \frac{n^2-\left(84\left(\frac{n-13}{14}\right)+57\right)}{7} + 20\left(\frac{n-13}{14}\right) + 15 = \frac{n^2+4n-4}{7}$ kings are needed.

$$\text{In summary, } \gamma_t\left(K_n\right) \leq \left\{ \begin{array}{c} \frac{n^2+4n-10}{7} \text{ for } n \equiv 12 \bmod 14 \\ \frac{n^2+4n-7}{7} \text{ for } n \equiv 3 \bmod 14 \\ \frac{n^2+4n-5}{7} \text{ for } n \equiv 8 \bmod 14 \\ \frac{n^2+4n-4}{7} \text{ for } n \equiv 4, 6, 13 \bmod 14 \\ \frac{n^2+4n-3}{7} \text{ for } n \equiv 5 \bmod 14 \\ \frac{n^2+4n}{7} \text{ for } n \equiv 0, 7, 10 \bmod 14 \\ \frac{n^2+4n+2}{7} \text{ for } n \equiv 1, 2, 9 \bmod 14 \\ \frac{n^2+4n+3}{7} \text{ for } n \equiv 11 \bmod 14 \end{array} \right\}.$$

In the long run, how does this upper bound on $\gamma_t\left(K_n\right)$ compare to $\gamma\left(K_n\right) \leq \gamma_t\left(K_n\right) \leq 2\gamma\left(K_n\right)$? We know $\gamma\left(K_n\right) = \left\lfloor \frac{n+2}{3} \right\rfloor^2$ and $\lim\limits_{n\to\infty} \frac{\frac{n^2+4n+c}{7}}{\left\lfloor \frac{n+2}{3} \right\rfloor^2} = 1.2857$. Over all, these constructions drive $\gamma_t\left(K_n\right)$ down from a possible maximum of $2\gamma\left(K_n\right)$ to $1.2857\gamma\left(K_n\right)$.

6 Future Work

With the two-dimensional cases worked out, the logical next step is to move into three dimensions. In fact, an analysis of the king's total domination number for the cube of side n was the original goal of our work. While studying the square of side n in preparation for the cube of side n, the flaw in Garnick and Nieuwejaar's upper bound appeared. The king's domination number for the square of side n, $\gamma\left(K_n\right) = \left\lfloor \frac{n+2}{3} \right\rfloor^2$, extends nicely to the king's domination number for the cube of side n, $\gamma\left(K_n^c\right) = \left\lfloor \frac{n+2}{3} \right\rfloor^3$. We hope to extend our own analysis into the third dimension and find a good bound for the total domination number. We have already begun investigating small cubes. The most notable result so far is that the total domination number for the cube of side 7 is the same as the domination number for the cube of side 7. By using Figure 6 on levels 2, 4 and 6 of the cube of side 7, $\gamma_t\left(K_7^c\right) \leq 27$. Since $\gamma\left(K_7^c\right) = 27$ we now know that $\gamma_t\left(K_7^c\right) = 27$.

	K			K	K	
	K				K	
		K				
	K			K	K	

Figure 6: Total Domination on Cube of Side 7

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