A Graph Theoretic Summation of the Cubes of the First n Integers

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The complete graph K_{n+1} contains $n+1$ vertices and $\binom{n+1}{2}$ edges. Iteratively building the complete graph K_{n+1} by introducing vertices one at a time and counting the new edges incident to the new vertex provides a combinatorial proof that $\sum_{i=1}^{n} i = \binom{n+1}{2}$.

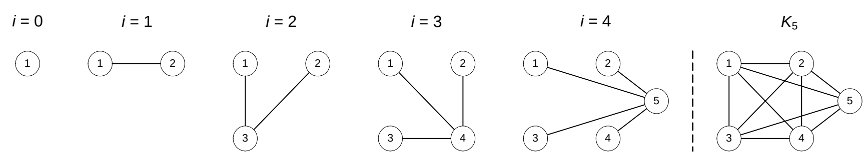
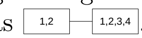


Figure 1: $\sum_{i=1}^{4} i = \binom{4+1}{2}$

Since $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$ it seems natural to look for a combinatorial proof that also uses graphs. Consider the complete bipartite graph $K_{\binom{n+1}{2},\binom{n+1}{2}}$ that contains $2\binom{n+1}{2}$ vertices and $\binom{n+1}{2}^2$ edges. As before, we will count the new edges incident to newly introduced vertices in n stages. At stage i we introduce i new vertices to each side of the graph and count the edges incident to these new vertices. Since $\sum_{i=1}^{n} i = \binom{n+1}{2}$ this process enumerates all the edges in $K_{\binom{n+1}{2},\binom{n+1}{2}}$. New vertices on one side are adjacent only to vertices on the other side. When just considering the edges between the new vertices, the subgraph $K_{i,i}$ immediately appears with i^2 edges. It turns out that these i^2 edges along with the additional edges constructed between a new vertex on one side and an old vertex on the other side will always total i^3 new edges. This shows that $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$.

In order to see that we always introduce i^3 new edges at stage i, we will partition the new edges into complete bipartite graphs. At stage i, there exist

$\binom{i}{2} = \frac{i(i-1)}{2}$ previously introduced vertices on each side of the graph and the new vertices on each side are labeled $\binom{i}{2} + 1, \binom{i}{2} + 2..., \binom{i}{2} + i = \binom{i+1}{2}$. The partition of these edges into complete bipartite graphs depends upon the parity of i. Figure 2 illustrates these stages for $n = 5$. To prevent a deluge of edges in the graph, a complete bipartite graph such as $K_{2,4}$ is represented as 

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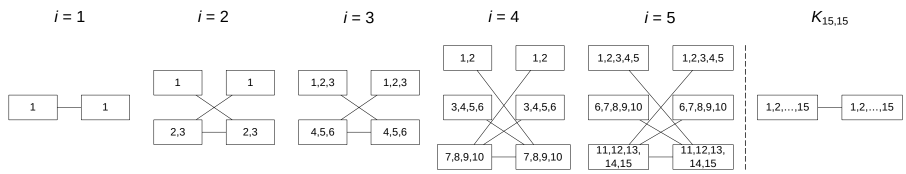

Figure 2: $\sum_{i=1}^{5} i^3 = \binom{5+1}{2}^2$

When i is odd, the new edges quickly form i disjoint copies of $K_{i,i}$. For odd i we partition the old vertices into $\frac{i-1}{2}$ sets of i vertices for each side. Both sets of i new vertices are adjacent to each of the $\frac{i-1}{2}$ sets of i vertices on the other side. This yields $2(\frac{i-1}{2}) = i - 1$ additional copies of $K_{i,i}$. Along with the initial copy of $K_{i,i}$ on only the new vertices, we have i copies of $K_{i,i}$ for a total of i^3 new edges.

When i is even, we have to work a bit harder. For even i, we partition the old vertices on each side into $\frac{i}{2} - 1$ sets of i vertices and one set of $\frac{i}{2}$ vertices. This yields $2\left(\frac{i}{2} - 1\right)$ copies of $K_{i,i}$ and two copies of $K_{\frac{i}{2},i}$ for $2\left(\frac{i}{2} - 1\right) i^2 + 2\frac{i}{2}i = i^3 - i^2$ edges. As before, with the original $K_{i,i}$ between the sets of new vertices, the total once again is i^3 new edges.

References

[1] J. DeMaio and J. Tyson, Proof without words: a graph theoretic summation of the first n integers, *The College Mathematics Journal* **38** (2007) 296.