Domination on the $m \times n$ Toroidal Chessboard by Rooks and Bishops

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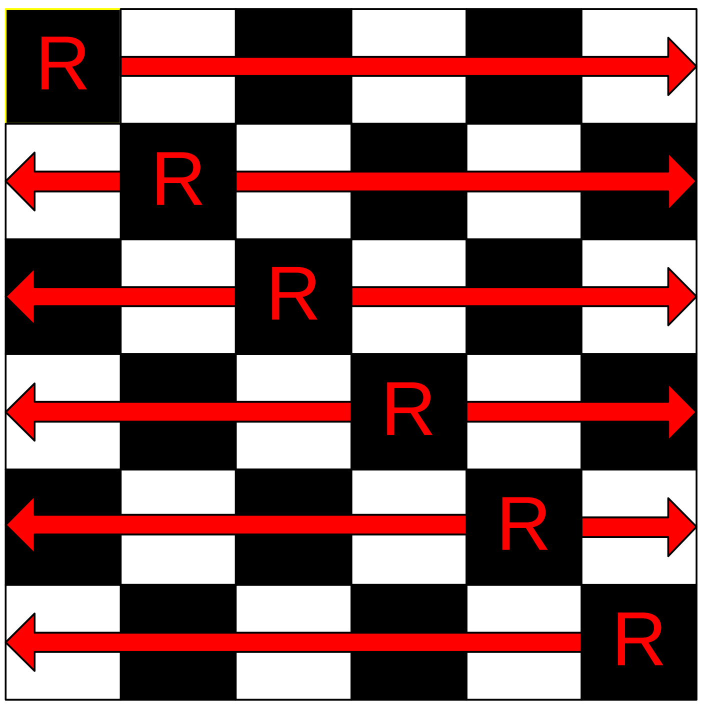
Domination

The **domination number**, γ, of a chessboard is defined as the minimum number of pieces that are needed to threaten or occupy every square on a chess board.



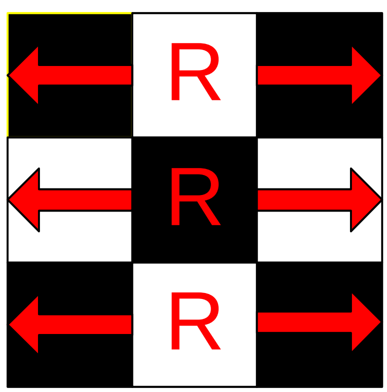
This is an example of a dominated chessboard with rooks. Notice that the movement of the rooks allow them to dominate any square within their row. (Similarly they can dominate any square within their column.)

Total Domination

The **total domination number**, γ_t, of a chessboard is defined as the minimum number of pieces that are needed to threaten every square on a chess board.

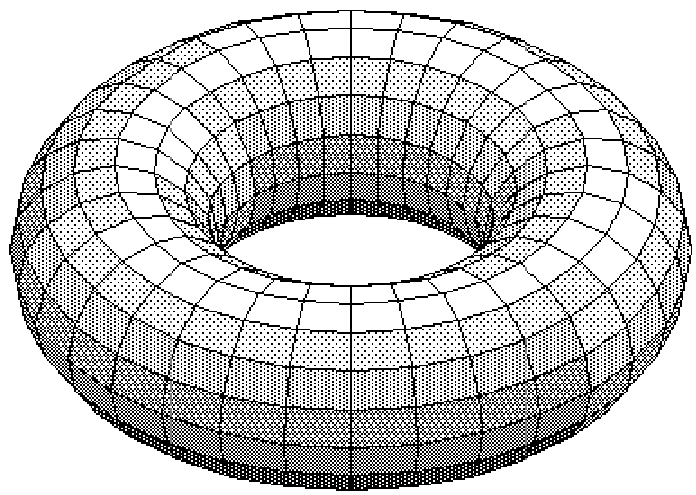
For total domination a piece no longer threatens the square it occupies.

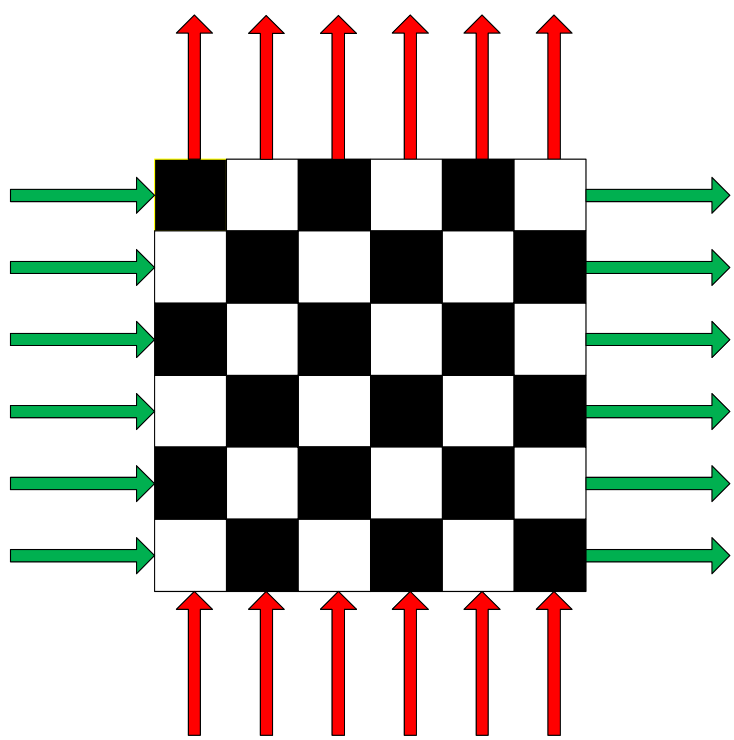
It is immediate that $\gamma \leq \gamma_t \leq 2\gamma$. A total dominating set is also a dominating set. For every vertex in a dominating set, a neighbor can be included to construct a total dominating set.



The rooks dominate the unoccupied squares as well as each other with their vertical movement.

The Torus

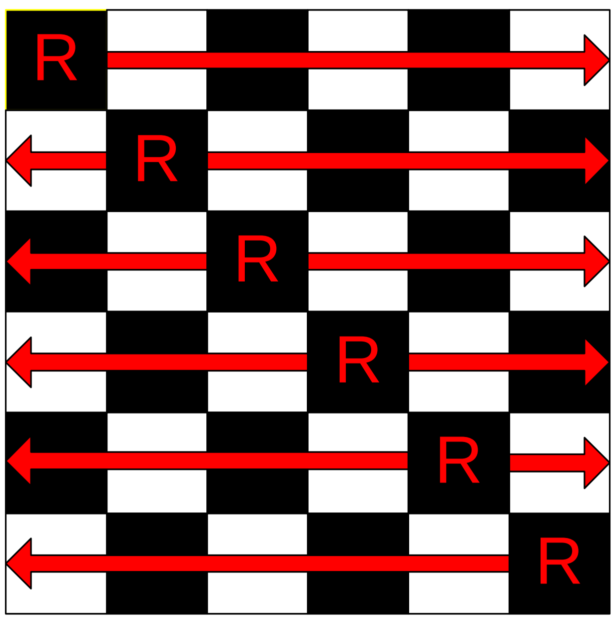




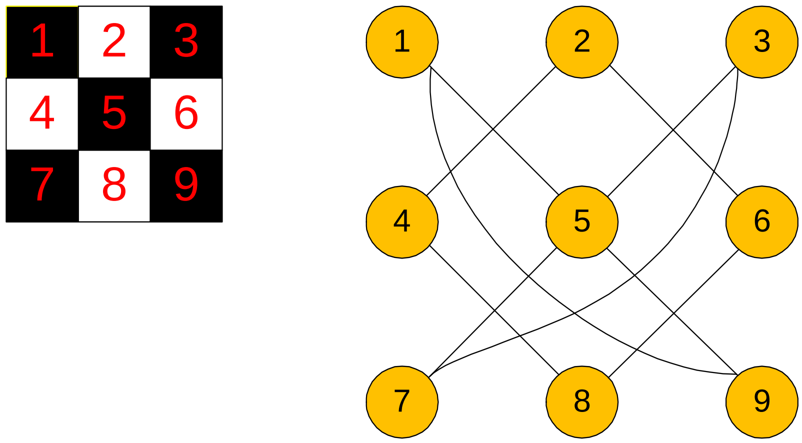
Rooks on the Torus

On the torus, the rook threatens no additional squares.

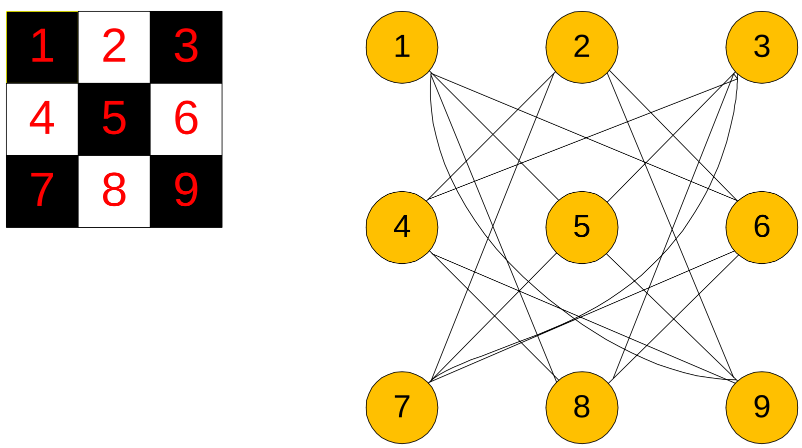
Thus, for $m, n \geq 2$, $\gamma(R^t_{m,n}) = \gamma_t(R^t_{m,n})$
$= \gamma(R_{m,n}) = \gamma_t(R_{m,n}) = \min\{m, n\}$.



Additional Freedom on the Torus for Bishops



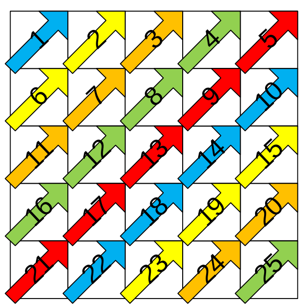
The figure above demonstrates how the movements of the bishop chesspiece on a two-dimensional chessboard can be transformed into a graph with vertices representing a bishop and each edge representing the ability to move to a different vertex.



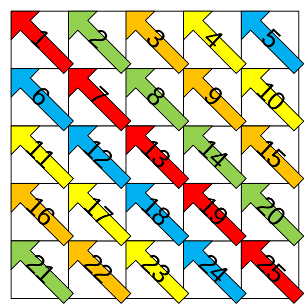
The figure above demonstrates how the movements of the bishop chess piece on the three-dimensional toroidal chessboard can be transformed into a graph. Notice that there are many more edges between the vertices compared to the graph of the two-dimensional chessboard.

Bishops Movement on the Torus

Since the torus has no borders, a bishop will always complete a cycle and return to its starting position from the opposite direction.



S-diagonals



D-diagonals

Theorem 1 : Bishops' moves are monochromatic if and only if both m and n are even.

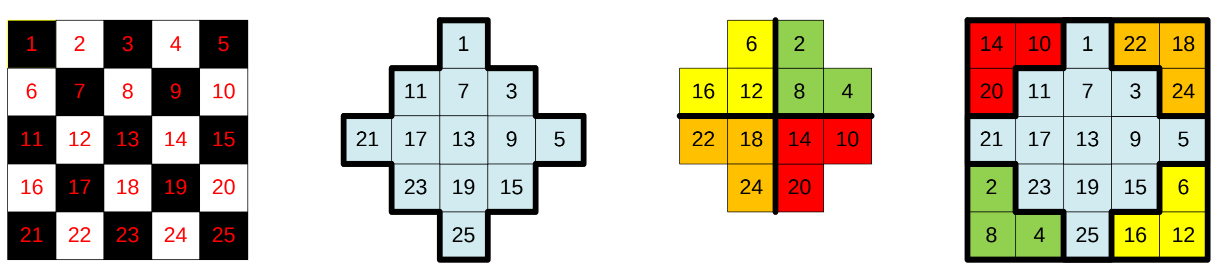
Theorem 2 : On the $m \times n$ rectangular torus, a bishop will attack $\mathrm{lcm}(m,n) = \frac{mn}{\gcd(m,n)}$ squares on either the s-diagonal or d-diagonal.

Bishop’s Domination Number on the Torus

Theorem 3 : For the rectangular $m \times n$ torus, $\gcd(m,n) = 1$ if and only if $\gamma(B^t_{m,n}) = 1$.

Theorem 4 : For the square torus of odd side n, $\gamma(B^t_n) = n$.

For odd n it is easy to redraw the bishops graph in the two-dimensional plane to resemble the rooks graph just as one does to analyze $\gamma(B_n)$.



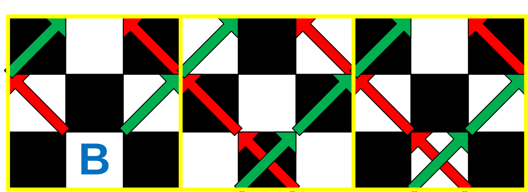
Quadrant board formation

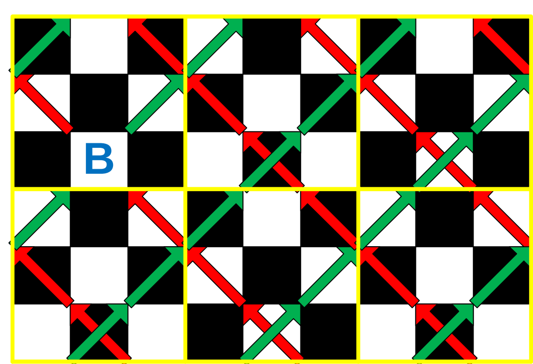
Theorem 5 : For the square torus of even side n, $\gamma(B_n^t) = n$.

For square boards of even side n it is not possible to redraw the bishop's graph in the two-dimensional plane to resemble the rook's graph.

Theorem 6 : For the rectangular $m \times km$ torus, $\gamma(B_{m,km}^t) = m$.

Theorem 7 : For the rectangular $jm \times km$ torus, $\gamma(B_{jm,km}^t) = m$ if and only if $\gcd(j,k) = 1$.



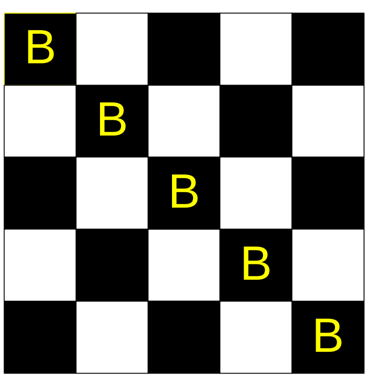


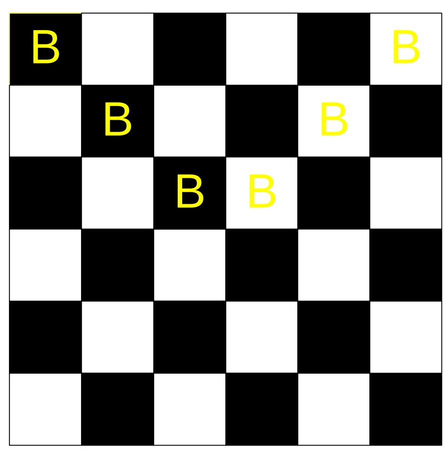
When working with bishops on the rectangular $m \times n$ torus, it is sufficient to focus on the initial square of side $\gcd(m, n)$. By combining the previous theorems together we achieve our main result.

Theorem 8 : For the rectangular $m \times n$ torus, $\gamma(B^t_{m,n}) = \gcd(m, n)$.

Bishop's Total Domination Number on the Torus

On the square torus of odd side $n \geq 3$, $\gamma_t(B_n^t) = n$. This is apparent since all bishops may be placed along the main diagonal of the board. On the square torus of even side $n \geq 4$, $\gamma_t(B_n^t) = n$. A different arrangement of bishops is needed since bishops are locked to squares of one color. In the even case it is necessary to place bishops on $\frac{n}{2}$ squares of both the main diagonal and the minor diagonal.





Consequences for the Queen's Domination Number on the Torus

Corollary 1 : For the rectangular $m \times n$ torus, if $\gcd(m,n) = 1$ then $\gamma(Q^t_{m,n}) = 1$.

Corollary 2 : For the rectangular $m \times n$ torus, if $\gcd(m,n) = 2$ and $n \geq 4$ then $\gamma(Q^t_{m,n}) = 2$.

Future Work

The following table gives the known and unknown values for graphs on both the two-dimensional chessboard and the toroidal chessboard:

Graph G	$ir(G)$	$\gamma(G)$	$i(G)$	$B_0(G)$	$\Gamma(G)$	$IR(G)$
R_n	n	n	n	n	n	$2n-4$
B_n	n	n	n	$2n-2$	$2n-2$	$4n-14$
$R_{m,n}$	?	$\min\{m,n\}$	$\min\{m,n\}$	$\min\{m,n\}$	?	?
$B_{m,n}$	?	?	?	?	?	?
$R^t_{m,n}$	?	$\min\{m,n\}$	$\min\{m,n\}$	$\min\{m,n\}$	?	?
$B^t_{m,n}$	?	$\gcd(m,n)$	$\gcd(m,n)$	$\gcd(m,n)$	?	?

Table 1: Domination Chain Values

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