Domination and Independence on the Rectangular Torus by Rooks and Bishops

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May 8, 2009

Abstract

A set $S \subseteq V$ is a dominating set of a graph $G = (V, E)$ if each vertex in V is either in S or is adjacent to a vertex in S. A vertex is said to dominate itself and all its neighbors. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is an independent set of vertices if no two vertices in S are adjacent. The independence number, $B_0(G)$, is the maximum cardinality of an independent set of G. Both $\gamma(G)$ and $B_0(G)$ are pieces of the six part domination chain: $ir(G) \leq \gamma(G) \leq i(G) \leq B_0(G) \leq \Gamma(G) \leq IR(G)$. Watkins has computed the domination numbers of rooks and bishops on the square torus. In this paper we compute the domination, total domination, independent domination and independence numbers of the bishop and rook on the rectangular $m \times n$ toroidal board.

Keywords: Graph, Domination, Independence, Chess

1 Introduction

Puzzles on the chessboard have long been studied by mathematicians. The survey papers Combinatorial Problems on Chessboards [7] and Combinatorial Problems on Chessboards II [9] provide excellent introductions to the various types of problems. Naturally, we do not restrict ourselves to the standard 8×8 chessboard. Generalizations are quickly made to the square board, the rectangular board, etc. A review of the literature, however, shows that very little work has been done to extend problems from the two-dimensional rectangular board to higher dimensions. Watkins computed domination numbers for rooks and bishops on the square torus [10]. In this paper we focus on the problems of domination and independence on the rectangular torus for rooks and bishops.

A set $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if each vertex in V is either in S or is adjacent to a vertex in S. A vertex is said to dominate itself and all its neighbors. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a *total dominating set* of a graph $G = (V, E)$ if each vertex in V is adjacent to a vertex in S. The *total domination number*, $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Since a total dominating set is a dominating set, $\gamma(G) \leq \gamma_t(G)$ for all graphs G. A set $S \subseteq V$ is an *independent set* of vertices if no two vertices in S are adjacent. The *independence number*, $B_0(G)$, is the maximum cardinality of an independent set of G. The *independent domination number*, $i(G)$, is the minimum cardinality of a maximal independent set of G. A maximal independent set of G must dominate G. If not then an undominated vertex could be included in the independent set. Note that $\gamma(G)$, $i(G)$ and

$B_0(G)$ are pieces of the six part domination chain, $ir(G) \leq \gamma(G) \leq i(G) \leq B_0(G) \leq \Gamma(G) \leq IR(G)$.

For further definitions, depth and detail in the study of the domination chain in graphs refer to [8].

In terms of a chessboard problem, let $X_{m,n}$ be the graph for chess piece X on the $m \times n$ board where $m \leq n$. We shall superscript $X_{m,n}$ with a t to indicate the graph for the $m \times n$ torus. Thus, $\gamma\left(B^t_{3,5}\right) = 1$ denotes that a single bishop threatens every square on the 3×5 torus while $\gamma\left(B^t_{3,6}\right) = 3$ indicates three bishops can threaten every square on the 3×6 torus but two bishops cannot threaten every square. The three-dimensional $m \times n$ torus is constructed from the two dimensional $m \times n$ board by connecting the right hand side of the board to left-hand side of the board (as one would construct a cylinder) and then proceed to connect the top of the board to the bottom as well. The resulting object resembles an inner tube or a donut.[1]

For the rook on the two-dimensional square board it is easy to show that $\gamma(R_n) = \gamma_t(R_n) = B_0(R_n) = n$ [12]. Values for the bishop are not quite as uniform and $\gamma(B_n) = n$ for all n [12], $\gamma_t(B_n) = 2\left\lceil \frac{2(n-1)}{3} \right\rceil$ for $n \geq 3$ [5] and $B_0(B_n) = 2n - 2$ [12]. It is trivial to extend these results for the rook to the two-dimensional $m \times n$ rectangular board and show that $\gamma(R_{m,n}) = \gamma_t(R_{m,n}) = B_0(R_{m,n}) = \min\{m, n\}$. It is non-trivial to extend these results for the bishop to the two-dimensional $m \times n$ rectangular board and your authors know of no published work on this topic.

Some work exists for the queen's graph on the torus in [1], [2] and [3] and this paper remains consistent with their terminology. Since the moves of the bishop are a subset of the moves of the queen, results for $\gamma\left(B^t_{m,n}\right)$ have direct implications on $\gamma\left(Q^t_{m,n}\right)$.[2]

2 Rooks on the Torus

Legal moves of the rook are easy to define on the torus. The rectangular board is wrapped into the form of a torus via the horizontal and vertical moves of the rook. On the torus the rook attacks no additional squares when compared to its rectangular board counterpart. For $m, n \geq 2$, $\gamma\left(R^t_{m,n}\right) = \gamma_t\left(R^t_{m,n}\right) = B_0\left(R^t_{m,n}\right) = \gamma(R_{m,n}) = \gamma_t(R_{m,n}) = B_0(R_{m,n}) = \min\{m, n\}$. Since $\gamma\left(R^t_{m,n}\right) = B_0\left(R^t_{m,n}\right) = \min\{m, n\}$, we squeeze the independent domination number $i\left(R^t_{m,n}\right) = \min\{m, n\}$ via the domination chain.

3 Movement of the Bishop on the Torus

The moves of the bishop on the torus are less obvious than the moves of the rook. We will formally construct the torus by starting with the standard $m \times n$ rectangular board with rows numbered $0, 1, ..., m-1$ and columns numbered $0, 1, ..., n-1$. Rows and columns of the board cycle around the torus. While the torus is a three-dimensional object, the cells are still referenced as an ordered pair. As described in [2] there are two diagonal moves for the bishop.[3] The northeast diagonal move of the bishop is the *sum diagonal*, or *s-diagonal*, since the sum of each coordinate pair on this line of attack is a fixed value k. Similarly, the northwest diagonal move of the bishop is the *difference diagonal*, or *d-diagonal*, since the positive difference of each coordinate pair on this line of attack is also a fixed value k. If two bishops threaten each other, we will call them *co-diagonal.* Otherwise we shall call them *non co-diagonal*.

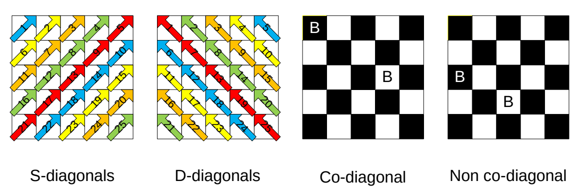

Figure 1: Toroidal Moves of the Bishop

[1] The latter, particularly so, if you think like Homer Simpson.

[2] Yes, the moves of the rook are also a subset of the moves of the queen. However, results for the rook are so trivial that no significant implications exist.

[3] Since the torus has no borders, a bishop will always complete a cycle and return to its starting position from the opposite direction. Hence, the two moves on the torus rather than four on the flat board.

On the torus any two non-collinear s-diagonals (or d-diagonals) are parallel and do not intersect. On the square torus of odd side n every bishop's s-diagonal and d-diagonal intersect only on the square occupied by the bishop. Furthermore, on the square torus of odd side n every s-diagonal of one bishop intersects the d-diagonal of every other bishop exactly once. For the torus of even side n, every bishop's s-diagonal and d-diagonal intersect on the square occupied by the bishop and one other square. Also on the torus of even side n when two bishops sit on different colors, no square is attacked by both bishops. In contrast to that, for any two bishops sitting on the same color, each s-diagonal of one bishop intersects the d-diagonal of the other bishop exactly twice. These intersecting lines of attack are discussed in depth (as the diagonal moves of the queen) in [2]. What happens as we extend the square torus to the rectangular torus? Bishops are locked to a color on the two-dimensional board. This is not always the case on the torus.

Theorem 1 *Bishops' moves are monochromatic if and only if both m and n are even.*

Proof. If either m (or n) is odd then the vertical (or horizontal) wrap of the torus will consist of two identical consecutive rows (or columns) which will force the bishop to alternate color. When both m and n are even, the wrap of the torus alternates rows identical to the flat two-dimensional board. ■

Theorem 2 *On the $m \times n$ rectangular torus, a bishop will attack* $\operatorname{lcm}(m,n) = \frac{mn}{\gcd(m,n)}$ *squares on either the s-diagonal or d-diagonal.*

Proof. Place a bishop on any (i, j) square on the torus. Dominating squares by moving one cell at a time on the s-diagonal will eventually return to the (i, j) square. On which move does the bishop first return to its starting position? Clearly the bishop returns to row i on move number sm for all $s \in Z^+$. Similarly, the bishop returns to column j on move number tn for all $t \in Z^+$. These will first coincide on move number $\operatorname{lcm}(m, n)$. This is, of course, also true for the d-diagonal. ■

4 Bishop's Domination Number on the Torus

As an immediate consequence of Theorem 2 we acquire a result for bishops on the torus that is dramatically different from bishops on the two-dimensional board!

Theorem 3 *For the rectangular $m \times n$ torus,* $\gcd(m, n) = 1$ *if and only if* $\gamma\left(B^t_{m,n}\right) = 1$.

Proof. Let $\gcd(m, n) = 1$. By Theorem 2, any bishop will threaten $\frac{nm}{\gcd(m,n)}$ squares on its s-diagonal. Since $\gcd(m, n) = 1$, a single bishop will dominate the entire torus and $\gamma\left(B^t_{m,n}\right) = 1$.

Now let $\gamma\left(B^t_{m,n}\right) = 1$. A single bishop dominates all mn squares. There are two possibilities. Either the bishop dominates all mn squares on the s-diagonal or both the s-diagonal and d-diagonal are needed to dominate all mn squares. If a single bishop dominates all mn squares on the s-diagonal then $\operatorname{lcm}(m, n) = mn$ which forces $\gcd(m, n) = 1$. If both the s-diagonal and d-diagonal are needed then $\operatorname{lcm}(m, n) = \frac{nm}{\gcd(m,n)} < nm$ and $\gcd(m, n) \geq 2$. If $\gcd(m, n) \geq 3$ then a bishop can dominate at most $\frac{2nm}{3}$ squares which is insufficient to dominate all mn squares of the torus. Thus, $\gcd(n, m) = 2$. However, this forces both m and n to be even. By Theorem 1 a single bishop on such a board will be locked to one color and be unable to dominate the entire torus. It is never the case that both the s-diagonal and d-diagonal are needed for a single bishop to dominate the entire torus and $\gcd(m, n) = 1$. ■

For those readers who are more interested in Hamiltonian tours [6], Theorem 3 quickly demonstrates the existence of a closed bishop's tour on the torus when $\gcd(m, n) = 1$.

On the square torus of side n, the cells of the board are partitioned by the n s-diagonals (or d-diagonals). Each diagonal wraps around the torus and contains exactly n squares. Thus, the bishop's graph on the square torus behaves much like the rooks graph. Just as with the square board, $\gamma\left(B^t_n\right) = n$ [10]. However, the cases behave differently for odd and even n.

Theorem 4 *For the square torus of odd side n, $\gamma\left(B_n^t\right) = n$.*

Proof. Since $\gamma\left(B_n\right) = n$ it is immediate that $\gamma\left(B_n^t\right) \leq n$. For odd n there are exactly n s-diagonals and n d-diagonals. Can we dominate the torus of side n with $n-1$ bishops? If so, then there will exist one s-diagonal without a bishop and one d-diagonal without a bishop. These two diagonals intersect in a square that is not dominated. Hence, $n-1$ bishops are insufficient and $\gamma\left(B_n^t\right) = n$.[4] ∎

For odd n it is easy to redraw the bishops graph in the plane to resemble the rooks graph just as one does to analyze $\gamma\left(B_n\right)$. Figure 2 demonstrates the construction for the square toroidal board of side 5.

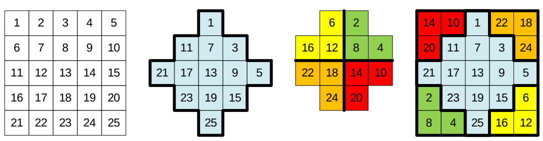

Figure 2: Transforming the Torus of Side 5 from Bishops to Rooks

Theorem 5 *For the square torus of even side n, $\gamma\left(B_n^t\right) = n$.*

Proof. Once again, since $\gamma\left(B_n\right) = n$ it is immediate that $\gamma\left(B_n^t\right) \leq n$. Of course, for even n, the rings of attack of a black bishop and a white bishop never overlap. Thus, we can partition the squares of the torus into their disjoint black and white boards which are identical. Applying the technique of Theorem 4, each disjoint monochromatic board requires $\frac{n}{2}$ bishops. Thus, the torus of even side n requires $\frac{n}{2} + \frac{n}{2} = n$ bishops. ∎

[4] Here is an esthetically pleasing alternative proof for non co-diagonal bishops on the torus of odd side n. Each bishop will occupy or threaten $2n-1$ squares. For odd n, two non co-diagonal bishops' rings of attack intersect twice. Consider $n-1$ pairwise non co-diagonal bishops. Each successively placed bishop dominates two fewer previously undominated squares than the previous bishop. With n bishops, the number of dominated squares is the sum of the first n odd integers. Thus, $\sum_{i=1}^{n} 2i-1 = n^2$ squares are dominated.

For even n it is not possible redraw the bishops graph in the plane to resemble the rooks graph since the number of intersecting squares for each non co-diagonal line is two.

Theorem 6 *For the rectangular $m \times km$ torus, $\gamma\left(B_{m,km}^t\right) = m$.*

Proof. A bishop on the (i,j) cell of the torus dominates the same cells as the bishop placed on the (r,s) cell where r and s are the least positive residues of i and j mod m. All k consecutive squares of side m that populate the $m \times km$ torus are identical with regard to the s-diagonals and d-diagonals of the bishops on the torus. Since it will take at least m bishops to dominate the m^2 cells of the first square of side m, $\gamma\left(B_{m,km}^t\right) \geq m$. Placing m bishops to form a dominating set on the first square of side m, dominates the entire $m \times km$ torus and $\gamma\left(B_{m,km}^t\right) = m$. ∎

Theorem 7 *For the rectangular $jm \times km$ torus, $\gamma\left(B_{jm,km}^t\right) = m$ if and only if $\gcd(j,k) = 1$.*
Proof. *Should $\gcd(j,k) = 1$ then each of the j horizontal semi-boards are identical when placing bishops for domination. If $\gcd(j,k) = d > 1$, then there exists a square torus of side md that will require md bishops to dominate.* ∎

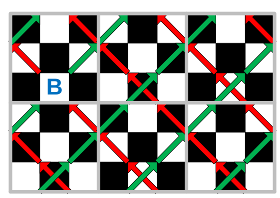

Figure 3: The 6×9 Rectangular Bishop Partitioned into Six Identical Squares of Side $\gcd(6,9) = 3$

When working with bishops on the rectangular $m \times n$ torus, it is sufficient to focus on the initial square of side $\gcd(n,m)$. Putting the previous theorems together we achieve our main result.

Theorem 8 *For the rectangular $m \times n$ torus,* $\gamma\left(B^t_{m,n}\right) = \gcd(m,n)$.

5 Bishop's Total Domination Number on the Torus

For the rectangular $m \times n$ torus, if $\gcd(m,n) = 1$ then $\gamma_t\left(B^t_{m,n}\right) = 2$. By Theorem 3, a single bishop will dominate all other squares on the toroidal board. The additional bishop is needed to attack the initial bishop. Similarly, if $\gcd(m,n) = 2$ then $\gamma_t\left(B^t_{m,n}\right) = 4$. Two bishops are sufficient to dominate the two disjoint monochromatic boards but two additional bishops are needed to attack the initial two bishops. For all $\gcd(m,n) \geq 3$, $\gamma\left(B^t_{m,n}\right) = \gamma_t\left(B^t_{m,n}\right)$. For all $\gcd(m,n)$ the entire torus is dominated by dominating the first square of side $\gcd(m,n)$. On the square torus of odd side $n \geq 3$, $\gamma_t\left(B^t_n\right) = n$. This is clear since all bishops may be placed on the main diagonal. On the square torus of even side $n \geq 4$, $\gamma_t\left(B^t_n\right) = n$. A different arrangement is needed since bishops are locked to a color. In the even case, place bishops on the top $\frac{n}{2}$ squares of both the main and minor diagonals.

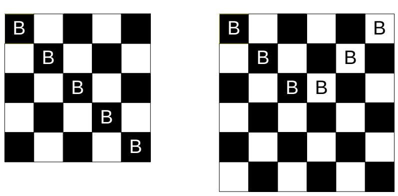

Figure 4: Total Domination by Bishops

6 Bishop's Independence Number on the Torus

Once again, the behavior of the initial square of side $\gcd(m,n)$ is replicated throughout the torus. There are $\gcd(m,n)$ s-diagonals (or d-diagonals) on the square torus of side $\gcd(m,n)$. Can we place $\gcd(m,n) + 1$ non-taking bishops in this square? No. By the pigeonhole principle, at least one s-diagonal (or d-diagonal) would contain at least two bishops that would threaten each other. Thus, $B_0\left(B^t_{m,n}\right) \leq \gcd(m,n)$. For both odd and even $\gcd(m,n)$ place all bishops in the first column of the initial square of side $\gcd(m,n)$. Hence, $B_0\left(B^t_{m,n}\right) = \gcd(m,n)$.

Since $\gamma\left(B^t_{m,n}\right) = B_0\left(B^t_{m,n}\right) = \gcd(m,n)$, we squeeze the independent domination number $i\left(B^t_{m,n}\right) = \gcd(m,n)$ via the domination chain.

7 Consequences for the Queen's Domination Number on the Torus

Since the moves of the bishop are a subset of the moves of the queen, it is immediate that $\gamma\left(Q^t_{m,n}\right) \leq \gamma\left(B^t_{m,n}\right)$ and our work can serve as an upper bound for the values of $\gamma\left(Q^t_{m,n}\right)$. Of particular interest is the fact that $\beta_0\left(Q^t_{m,n}\right) = \gcd(n,m)$ except for $m = 3$ and $n = 6$ [3]. As part of the domination chain it is well known that $\gamma(G) \leq \beta_0(G)$. The implications of this inequality are not exploited in [3]. For starters if $\gcd(m,n) = 1$ then then domination number must also be 1 since $\gamma\left(Q^t_{m,n}\right) \leq \beta_0\left(Q^t_{m,n}\right) = 1$. As an immediate consequence of Theorem 3, we get the following corollary as an alternative proof.

Corollary 1 *For the rectangular $m \times n$ torus, if* $\gcd(m,n) = 1$ *then* $\gamma\left(Q^t_{m,n}\right) = 1$.

Proof. Clearly, $\gamma\left(Q^t_{m,n}\right) \leq \gamma\left(B^t_{m,n}\right) = 1$ when $\gcd(m,n) = 1$. ■

Unlike Theorem 3, the queen's version of this theorem is not an if and only if since $\gamma\left(Q^t_2\right) = \gamma\left(Q^t_3\right) = 1$.

Theorem 9 *For the rectangular $m \times n$ torus, if* $\gcd(m,n) = 2$ *and* $n \geq 4$ *then* $\gamma\left(Q^t_{m,n}\right) = 2$.

Proof. Like the bishop, the diagonal moves of the queen are color locked on the torus if both m and n are even. Of course, the queen can attack squares of opposite color (than its original position) by its horizontal and vertical lines of attack. However, if $n \geq 4$ then there exists a knight's move from the queen to a square of opposite color that does not fall on the queen's horizontal or vertical attack lines. Thus, a single queen cannot dominate the entire torus and $\gamma\left(Q^t_{m,n}\right) > 1$. However, if $\gcd(n, m) = 2$ then $\gamma\left(Q^t_{m,n}\right) \leq \gamma\left(B^t_{m,n}\right) = 2$. Hence, $\gamma\left(Q^t_{m,n}\right) = 2$. ■

8 Future Work

Values in the domination chain for the bishop's and rook's graphs on the square of side n are all known. Work on the values in the domination chain for the bishop's graph on the rectangular two-dimensional board should begin and values for the rook should be completed. The remaining values in the domination chain for the bishop's and rook's graphs on the torus should be determined. On the torus, bishops behave so much like rooks that it appears that determining the remaining values of the domination chain will be much easier than working with a very irregular (as bishops move), two-dimensional rectangular board.

Graph G	$ir(G)$	$\gamma(G)$	$i(G)$	$B_0(G)$	$\Gamma(G)$	$IR(G)$
R_n	n	n	n	n	n	$2n-4$
B_n	n	n	n	$2n-2$	$2n-2$	$4n-14$
$R_{m,n}$	?	$\min\{m,n\}$	$\min\{m,n\}$	$\min\{m,n\}$	?	?
$B_{m,n}$	?	?	?	?	?	?
$R^t_{m,n}$	?	$\min\{m,n\}$	$\min\{m,n\}$	$\min\{m,n\}$	?	?
$B^t_{m,n}$	?	$\gcd(m,n)$	$\gcd(m,n)$	$\gcd(m,n)$	?	?

Table 1: Domination Chain Values

9 Acknowledgements

The authors would like to thank the Mentor-Protégé Undergraduate Research Program in the College of Science & Mathematics at Kennesaw State University for funding our project.

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