with $\beta_0 < \beta_1 < \cdots < \beta_n$ and $b_n \neq 0$, has more than n roots. This contradiction to the induction hypothesis concludes the proof. ■

The Exponential Theorem generalizes the fundamental theorem of algebra to exponential functions the way the Generalized Polynomial Theorem does for generalized polynomials. A striking fact is that the two proofs follow the same path. Despite appearances, the theorems are equivalent, as the following argument shows.

Let $f(t) = \sum_{j=0}^{n} a_j \kappa_j^t$ with $0 < \kappa_0 < \kappa_1 < \cdots < \kappa_n$, $a_j \in \mathbb{R}$, and $a_n \neq 0$. Let $\kappa_0 = e^{c_0}, \kappa_1 = e^{c_1}, \ldots, \kappa_n = e^{c_n}$ for some $c_0 < c_1 < \cdots < c_n$. By multiplying $f(t)$ by Δ^t for a suitable $\Delta > 1$, we may assume that $c_0 > 0$ to ascertain that $1 < c_1/c_0 < \cdots < c_n/c_0$. Then

$$\begin{aligned} f(t) &= a_0 e^{c_0 t} + a_1 e^{c_1 t} + \cdots + a_n e^{c_n t} \\ &= a_0 e^{c_0 t} + a_1 \left(e^{c_0 t}\right)^{c_1/c_0} + \cdots + a_n \left(e^{c_0 t}\right)^{c_n/c_0} \\ &= a_0 x + a_1 x^{c_1/c_0} + \cdots + a_n x^{c_n/c_0} = g(x) \end{aligned}$$

with $x = e^{c_0 t}$. By the Generalized Polynomial Theorem, with $\alpha_0 = 1, \alpha_1 = c_1/c_0, \ldots, \alpha_n = c_n/c_0$, there exist at most k positive roots of the corresponding function $g(x)$. Certainly, when x_i is such a root, $t_i := (\ln x_i)/c_0$ becomes a root of $f(t)$, and vice-versa. This way, we have shown that the Generalized Polynomial Theorem implies the Exponential Theorem. The opposite implication comes from reversing the argument.

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Closed Knight's Tours with Minimal Square Removal for All Rectangular Boards

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Finding a closed knight's tour of a chessboard is a classic problem: Can a knight use legal moves to visit every square on the board and return to its starting position? [**1, 3**] An open knight's tour is a knight's tour of every square that does not return to its starting position. While originally studied for the standard 8×8 board, the problem is easily generalized to other rectangular boards. In 1991 Schwenk classified all rectangular boards that admit a closed knight's tour [**2**]. He described every board that cannot admit a closed knight's tour and constructed closed knight's tours for all other boards.

SCHWENK'S THEOREM. *An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour unless one or more of the following three conditions hold:*

(a) *m and n are both odd;*

(b) $m \in \{1, 2, 4\}$;

(c) $m = 3$ *and* $n \in \{4, 6, 8\}$.

How close to admitting a closed knight's tour are those boards that satisfy Schwenk's conditions (a), (b), or (c)? Let us call these *obstructed boards*, since they admit no closed knight's tour. The 3×3 board is obstructed; however, once the center square is removed a closed knight's tour does indeed exist as seen in FIGURE 1.

1	6	3
4		8
7	2	5

Figure 1 A closed knight's tour of the 3×3 board with the center square removed

Let the *tour number*, $T(m, n)$, with $m \leq n$ be the minimum number of squares whose removal from an $m \times n$ chessboard will allow a closed knight's tour. Thus, $T(3, 3) = 1$. Note that unless m and n are the dimensions of an obstructed board, $T(m, n) = 0$; no squares need to be removed. Also note that removing $T(m, n)$ squares randomly from an obstructed board does not guarantee the existence of a closed knight's tour. For instance, removing any square other than the center does not allow for a closed knight's tour of the resulting 3×3 board. Furthermore $T(1, n)$ and $T(2, 2)$ are undefined since the knight cannot move from its starting position. Also $T(2, n) = 2n - 2$ for $n \geq 3$ since a knight can move down a $2 \times n$ board but cannot return to its starting position unless only one move has been made.

Parity considerations restrict the number of squares we can remove from an obstructed board if we hope to get a closed knight's tour. Throughout this paper, whenever we color the squares of a chessboard black and white, we will always begin with a black square in the upper left-hand corner. A legal move for a knight whose initial position is a white square will always result in an ending position on a black square and vice versa. Hence, any closed knight's tour must visit an equal number of black squares and white squares. This quickly determines that an odd number of squares must be removed from a board where both m and n are odd and an even number of squares must be removed from all other boards.

Thus, for an obstructed board, the smallest possible tour numbers are 1 and 2 respectively for boards with an odd or even number of squares. Recursive constructions of closed knight's tours will show that tour numbers are actually as small as possible, except in a few special cases. The constructions start with small boards, called *base boards*, and build by tacking on boards with open tours.

We compute all nonzero tour numbers by considering the three cases from Schwenk's Theorem: $m = 3$, $m = 4$, and the case where m and n are both odd.

The case of $m = 3$ For odd n, two base boards are needed for $n \equiv 1, 3 \mod 4$ and one exceptional case exists for $n = 5$. For even n, three boards are examined for $n = 4, 6,$ and 8.

To construct a closed knight's tour for the 3×7 board, start with the open 3×4 tour from FIGURE 2, which begins at a and ends at l. Next, take the tour for the 3×3 board

a	*d*	*g*	*j*
l	*i*	*b*	*e*
c	*f*	*k*	*h*

Figure 2 An open knight's tour of the 3×4 board

with the center square removed as in FIGURE 1 and delete the 5–6 move. Connect the boards by creating the 5–a and 6–l moves (these correspond to legal knight moves) as shown in FIGURE 3. This is typical of our approach throughout the paper.

1	6	3	a	d	g	j
4		8	l	i	b	e
7	2	5	c	f	k	h

Figure 3 A closed knight's tour of the 3×7 board after minimal square removal

This game can be played an infinite number of times replacing the role of squares 5 and 6 by g and h. Thus, $T(3, n) = 1$ for all $n \equiv 3 \bmod 4$. Note that the lower right hand corner of any board, when used, must contain the g–h move as there are only two legal moves for a knight from that corner square.

For the 3×5 board note that the corner squares have only two legal moves, where one move is the center square. At most two of these four squares may be included in a tour. Hence at least two of these squares must be removed. Furthermore, all four of those corner squares are black and it will also be necessary to remove at least one white square. Thus, $T(3, 5) \geq 3$. The existence of the tour in FIGURE 4 shows that $T(3, 5) = 3$.

1	4	7	10	
	9	12	3	6
	2	5	8	11

Figure 4 A closed knight's tour of the 3×5 board after minimal square removal

But the 3×5 board is the lone exception for all boards with an odd number of squares. FIGURE 5 shows that $T(3, 9) = 1$. As before, the open tour of FIGURE 2 yields $T(3, n) = 1$ for all $n \equiv 1 \bmod 4$ where $n \neq 5$.

1	4	7	18	21	24	9	12	15
6	19	2	25	8	17	14	23	10
3	26	5	20		22	11	16	13

Figure 5 A closed knight's tour of the 3×9 board after minimal square removal

The case of the 3×4 board is a very straightforward one. As shown in FIGURE 6, if two squares are removed, a knight's tour exists. Thus, $T(3, 4) \leq 2$. Since an even number of squares is required, $T(3, 4) = 2$. Furthermore, tacking on the open tour of FIGURE 2 shows that $T(3, 8) = 2$.

1	4	9	6
	7	2	
3	10	5	8

Figure 6 A closed knight's tour of the 3×4 board after minimal square removal

The 3×6 board is the only other obstructed one with $m = 3$. We will analyze FIGURE 7 to show that $T(3, 6) = 4$.

1	4	7	10	13	16
2	5	8	11	14	17
3	6	9	12	15	18

Figure 7 A way to label the 3×6 board

Using all four corners immediately forces the paths 6–1–8–3–4 and 13–18–11–16–15. Since squares 8 and 11 can have no further connections, these paths are necessarily extended to 7–6–1–8–3–4–9 and 12–13–18–11–16–15–10. However none of the four remaining squares (2, 5, 14, and 17) can be included without closing one of these paths before connecting to the other. No tour exists using all four corners that omits exactly two squares. Furthermore, including the 7–12 and 9–10 moves creates a tour when omitting 4 squares and $T(3, 6) \leq 4$.

Next note that squares 5 and 17 cannot both be used without creating a closed cycle with 10 and 12. Similarly, squares 2 and 14 cannot both be used without creating a closed cycle with 7 and 9. When combined with the previous fact that no tour exists using all four corners that omits exactly two squares we achieve $T(3, 6) \geq 3$. Since $T(3, 6)$ is even, $T(3, 6) = 4$.

The case of $m = 4$ Any board with $m = 4$ will have an even number of squares; thus, $T(4, n)$ will always be even. The boards of FIGURE 8 show that $T(4, 4) = T(4, 5) = T(4, 6) = 2$.

	5	10	1
13	2	7	4
6	9	14	11
	12	3	8

12	3	16	7	10
17	8	11	2	15
4	13	18	9	6
		5	14	1

21	2	13	8	17	4
12	7	22	3	14	9
1	20	11	16	5	18
		6	19	10	15

Figure 8 Closed knight's tours of the 4×4, 4×5, and 4×6 boards after minimal square removal

Next consider any tour on a $4 \times k$ board that contains the a–b and c–d moves in the lower right-hand corner as in the left-hand side of FIGURE 9. This $4 \times k$ tour can be extended to a $4 \times (k+3)$ tour by removing moves a–b and c–d and connecting to a 4×3 extension with the moves 1–c, 6–d, a–a_1, and b–a_6. Note that all three base boards and the 4×3 extension contain the a–b and c–d moves in the lower right-hand corner as in FIGURE 9. This proves $T(4, n) = 2$ for all $n \geq 4$.

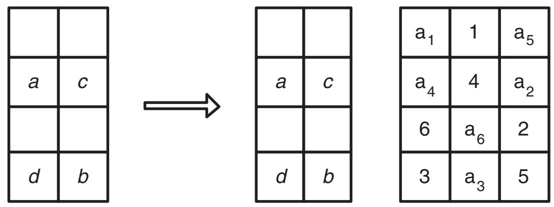

Figure 9 A closed knight's tour of the $4 \times (n+3)$ board from a closed knight's tour of the $4 \times n$ board

The case of both *m* and *n* odd Much like the $m = 3$ and $m = 4$ cases, we use induction with an appropriate base case to analyze all boards with an odd number of squares for $m \geq 5$. In all cases, $T(m, n) = 1$ for both m and n odd with $5 \leq m \leq n$. Four base cases exist, one for each combination of $m, n \equiv 1, 3 \bmod 4$. The boards of FIGURE 10 are used for $m, n \equiv 1 \bmod 4$ and $m \equiv 1, n \equiv 3 \bmod 4$ respectively. For $m, n \equiv 3 \bmod 4$ use the 3×7 board of FIGURE 3 and for $m \equiv 3, n \equiv 1 \bmod 4$ use the 3×9 board of FIGURE 5. The open 3×4 tour of FIGURE 2 and the open 5×4 tour of FIGURE 11 can be used to extend the base boards to any length $n \equiv 1, 3 \bmod 4$ as demonstrated in FIGURE 3. For the 5×5 board delete the 10–11 move and create the 1–10 and 11–20 moves. For the 5×7 board delete the 26–27 move and create the 1–16 and 20–27 moves.

1	18	7	12	
8	13	24	17	22
19	2	21	6	11
14	9	4	23	16
3	20	15	10	5

1	30	17	8	23	28	15
18	9	34	29	16	7	24
31	2		22	25	14	27
10	19	4	33	12	21	6
3	32	11	20	5	26	13

Figure 10 Base boards for $m \equiv 1 \bmod 4$

We have constructed tours for all $3 \times n$ and $5 \times n$ boards for odd $n \geq 7$. Next we need to extend these boards down to an arbitrary odd m. To do so, rotate clockwise the open tours of FIGURE 2 and FIGURE 11 to a 4×3 tour and a 4×5 tour and extend the base $3 \times n$ and $5 \times n$ boards down to any depth $m \equiv 1, 3 \bmod 4$. For $m, n \equiv 1 \bmod 4$ use the $5 \times n$ board (created with FIGURE 10) delete the 14–15 move and create the 1–14 and 15–20 moves with FIGURE 11 rotated clockwise. For $m \equiv 1, n \equiv 3 \bmod 4$ use the $5 \times n$ board (created with FIGURE 10), delete the 3–4 move and create the 3–a and 4–l moves with FIGURE 2 rotated clockwise. For $m \equiv 3, n \equiv 1 \bmod 4$ use the $3 \times n$ board (created with FIGURE 5), delete 5–6 move and create the 1–6 and 5–20 moves with FIGURE 11 rotated clockwise. For $m, n \equiv 3 \bmod 4$ use $3 \times n$ board

12	17	8	3
7	2	13	18
16	11	4	9
1	6	19	14
20	15	10	5

Figure 11 An open tour of the 5×4 board

(created with FIGURE 3), delete the 7–8 move and create the 7–a and 8–l moves with FIGURE 2 rotated clockwise. This process provides us with a closed knight's tour of the top and left side of the board in FIGURE 12.

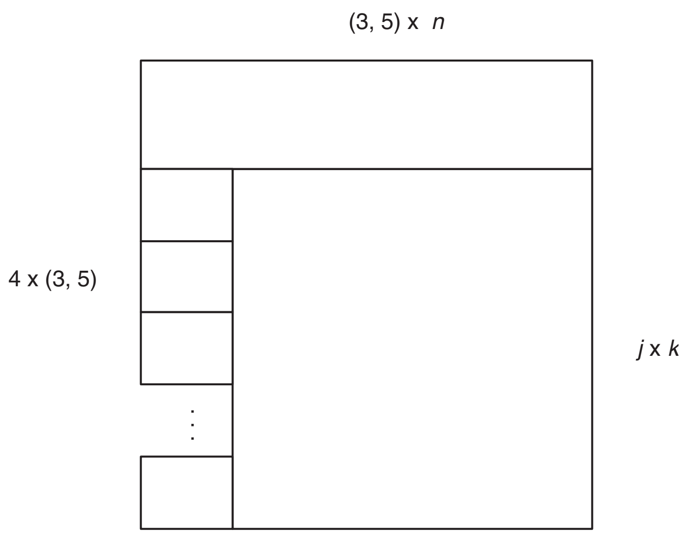

Figure 12 Constructing a closed knight's tour of the $m \times n$ board after minimal square removal for $m, n \equiv 1 \bmod 2$

Now a $j \times k$ gap with $j, k \equiv 0 \bmod 4$ needs to be filled in to complete the $m \times n$ board. Finally, we use the 4×4 board of FIGURE 13 to fill in the $j \times k$ gap using the same technique of FIGURE 9.

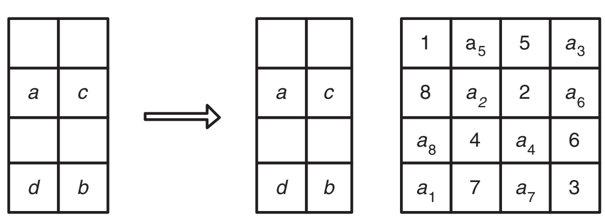

Figure 13 Filling in a $j \times k$ gap for $j, k \equiv 0 \bmod 4$

Conclusion In summary, the tour number for obstructed boards is as small as possible (1 or 2) based on an odd or even number of squares with the few noted exceptions as indicated below.

For the $m \times n$ chessboard with $m \leq n$, either board has a closed knight's tour, so that $T(m, n) = 0$, or else

(a) $T(m, n) = 1$, where m and n are both odd except for $m = 3$ and $n = 5$;
(b) $T(4, n) = 2$ for all $n \geq 4$;
(c) $T(3, 4) = T(3, 8) = 2$, $T(3, 5) = 3$, $T(3, 6) = 4$;
(d) $T(2, n) = 2n - 2$ for $n \geq 3$;
(e) $T(1, n)$ and $T(2, 2)$ are undefined.

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