Closed Monochromatic Bishops' Tours

Joe DeMaio

Department of Mathematics and Statistics

Kennesaw State University, Kennesaw, Georgia, 30144, USA

jdemaio@kennesaw.edu

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Abstract

In chess, the bishop is unique as it is locked to a single color on the black and white board. This makes a closed tour in which the bishop visits every square on the board exactly once and returns to its starting position impossible. When can two bishops, one black and one white, legally visit every square (of their respective colors) exactly once and return to their starting positions? Such a tour will be called a *closed monochromatic bishop's tour*. In this paper necessary and sufficient conditions for the existence of a monochromatic bishop's tour for the rectangular $m \times n$ board are proven. Furthermore, a monochromatic knight's move is defined for the three dimensional chessboard and a closed monochromatic knight's tour is provided for the cube of side 6.

1 Introduction

Puzzles on the chessboard have long been studied by mathematicians. The survey papers Combinatorial Problems on Chessboards [2] and Combinatorial Problems on Chessboards II [3] provide excellent introductions to the various types of problems. The closed knight's tour is one of these much studied and famous problems in the area of chessboard puzzles. For which boards can a knight legally visit every square exactly once and return to its starting position? Schwenk completely answered this question for rectangular boards in 1991.

Theorem 1 *An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour unless one or more of the following three conditions hold:*

(a) m and n are both odd;
(b) $m \in \{1, 2, 4\}$;
(c) $m = 3$ and $n \in \{4, 6, 8\}$.

The unique movement of the knight makes for interesting study while closed tours for the king, queen and rook are trivial to construct. The bishop is color locked and clearly cannot tour every square on the board. Let's change the question slightly. When can two bishops, one black and one white, legally visit every square (of their respective colors) exactly once and return to their starting positions? Such a tour will be called a *closed monochromatic bishops' tour.* Occasionally we will need a monochromatic tour that visits every square exactly once but does not return to its starting position. Such a tour is an *open monochromatic bishops' tour.* For this paper we will assume the top left square is always black in an initial $m \times n$ chessboard with $m \leq n$.

2 The Case of $m = 1, 2$

For the $1 \times n$ chessboard, the bishops are unable to make a single move and no closed monochromatic bishops' tour exists. For the $2 \times n$ chessboard the bishops are able to move down the board but are unable to return without repeating a square. Thus, except for the 2×2 chessboard, no closed monochromatic bishops' tour exists for $m = 2$.

3 The Case of $m = 3$

No closed monochromatic bishops' tour exists for $m = n = 3$. While a tour of the white squares is possible (and easy), the black squares prove to be troublemakers. The four corner black squares each have only two possible moves, one of which is to the center square. For a tour to exist each corner must be preceded or succeeded by the center square. This forces the center square to be visited more than once and no closed monochromatic bishops' tour exists.

Given the 3×4 board below, we can construct any length $n \equiv 0 \mod 4$ open monochromatic bishops' tour by placing copies of this board side by side. Connect the black paths by moving from cell 4 of the left board to cell 1 of the right board and from cell b of the left board to cell a of the right board. The same moves will also connect the white paths. It is simple to connect the two black paths on the rightmost side by constructing the $4 - b$ edge. Connecting the two white paths requires a bit more. First, delete the $2 - 3$ edge and create the $2 - 4$ and $3 - b$ edges. We are now left with an open monochromatic bishops' tour with ends on the leftmost side of the board at 1 and a for both the black and white tours.

Black

1		*b*	
	2		4
a		3	

White

	2		*b*
1		3	
	a		4

Figure 1: The 3×4 Board to Create the $3 \times 4k$ Open Tour

To create a closed tour for any $n \mod 4$, prepend the appropriate $3 \times r$ board for $r \equiv n \mod 4$ and follow the chart to close off the open ends of the tour.

x
y
z

r	*x*
s	*y*
t	*z*

i	*r*	*x*
j	*s*	*y*
k	*t*	*z*

Figure 2: Boards to Prepend to Open $3 \times n$ Tour

$n \mod 4$	Delete Black	Create Black
0	$2-3$	$1-3, 2-a$
1	none	$1-y, a-y$
2	none	$1-t,\ t-y,\ r-y,\ a-r$
3	none	$1-y,\ t-y,\ j-t,\ j-r,\ a-r$
$n \mod 4$	Delete White	Create White
0	none	$1-a$
1	$1-2$	$1-x, 1-z, 2-z, a-x$
2	none	$1-z,\ s-z,\ s-x,\ a-x$
3	none	$1-z,\ i-z,\ i-s,\ s-k,\ k-x,\ a-x$

Each $3 \times n$ board can be extended to a $m \times n$ board for $m \equiv 0 \mod 3$. Note that every $3 \times n$ board's rightmost side is identical. Create two $3 \times n$ boards placed lengthwise. For both black and white tours delete the rightmost $3-4$ edges in the top tours and the rightmost $1-2$ edges in the bottom tours. Now create the $1-3$ and $2-4$ edges. This yields a construction for all closed monochromatic bishops' tours for the $m \times n$ board for $m \equiv 0 \mod 3$ and $n \geq 4$.

4 The Case of $m = 4$

1	*f*	3	*h*
c	2	*g*	6
4	*b*	7	*e*
a	5	*d*	8

1	*e*	3	*i*	6
d	2	*j*	7	*h*
4	*b*	10	*g*	8
a	5	*c*	9	*f*

1	*e*	3	*k*	6	*h*
d	2	*l*	7	*i*	10
4	*b*	12	*g*	9	*j*
a	5	*c*	11	*f*	8

1	*e*	3	*m*	6	*h*	12
d	2	*n*	9	*j*	11	*g*
4	*b*	14	*i*	10	*k*	7
a	5	*c*	13	*f*	8	*l*

1	*e*	3	*o*	6	*l*	10	*h*
d	2	*p*	13	*n*	9	*g*	11
4	*b*	16	*m*	14	*i*	7	*k*
a	5	*c*	15	*f*	12	*j*	8

Figure 3: Base Cases for $m = 4$

Figure 3 provides us with five different closed monochromatic tours of length 4 and varying widths. As with the $3 \times n$ boards, these base cases can easily extend to a closed monochromatic bishops tour for any $4 \times n$ board. The chart below indicates how to use the closed monochromatic bishops tour of the 4×4 board to create a closed monochromatic bishops tour for any $4 \times n$ board.

n mod 4	Delete Black	Create Black
0	$7-8$ in left $4 \times n, 1-2$ in right 4×4	$1-7, 2-8$
1	$6-7$ in left $4 \times n, a-b$ in right 4×4	$6-b, 7-a$
2	$8-9$ in left $4 \times n, 1-2$ in right 4×4	$1-9, 2-8$
3	$11-12$ in left $4 \times n, a-b$ in right 4×4	$11-a, 12-h$
n mod 4	Delete White	Create White
0	$g-h$ in left $4 \times n, a-b$ in right 4×4	$a-g, b-h$
1	$f-g$ in left $4 \times n, 1-2$ in right 4×4	$1-g, 2-f$
2	$h-i$ in left $4 \times n, a-b$ in right 4×4	$a-i, b-h$
3	$k-l$ in left $4 \times n, 1-2$ in right 4×4	$1-k, 2-l$

Just like the $3 \times n$ boards, we can extend these $4 \times n$ boards to any $m \times n$ board for $m \equiv 0 \mod 4$ and $n \geq 4$ except for the 4×4 board. This necessitated the inclusion of the 4×8 base case. Stack two boards top to bottom as before. For the black bishop, delete $1-2$ in the leftmost side of the bottom board. In the top board, delete in the leftmost side the $9-10$, $11-12$, $13-14$ or $15-16$ (depending upon the base case) edge. Next construct the $1-10$ and $2-9$ edges (or other edges depending upon the base case) to extend the black bishops' tour. For the white bishop, remove the $a-b$ edge from the leftmost top board and the $i-j$, $k-l$, $m-n$ or $o-p$ (depending upon the base case) edge. Next construct the $a-j$ and $b-i$ edges (or other edges depending upon the base case) to extend the white bishops' tour. This construction would not work on the 4×4 board as we would need to delete the $7-8$ edge twice.

5 Frobenius Combinations

If we can attach these $3 \times n$ and $4 \times n$ tours to each other then any $m \times n$ tour for $m \geq 6$ can be constructed from the base boards as the Frobenius number $g(3, 4) = (3-1)(4-1) = 6$. Fortunately, this is easy to accomplish. Place the appropriate number of copies of the $4 \times n$ tours lengthwise followed by the appropriate number of $3 \times n$ tours. On the rightmost side of the boards for the bottommost $4 \times n$ board and uppermost $3 \times n$ board follow the chart to combine the tours. Note that the color of the squares for the $3 \times n$ board will switch based on $n \mod 2$.

$n \mod 4$	Delete Black	Create Black
0	$1-2$ in $3 \times n$, $15-16$ in $4 \times n$	$1-15, 2-16$
1	$2-4$[1]in $3 \times n$, $4-5$in $4 \times n$	$2-5, 4-4$
2	$1-2$ in $3 \times n$, $4-5$ in $4 \times n$	$1-4, 2-5$
3	$2-4$ in $3 \times n, 13-14$ in $4 \times n$	$2-13$, $4-14$
$n \mod 4$	Delete White	Create White
0	$2-4$ in $3 \times n$, $e-f$ in $4 \times n$	$2-e, 4-f$
1	$3-4$ in $3 \times n$, $a-b$ in $4 \times n$	$3-a, 4-b$
2	$2-4$ in $3 \times n, b-c$ in $4 \times n$	$2-b, 4-c$
3	$1-2$ in $3 \times n, b-c$ in $4 \times n$	$1-b, 2-c$

6 The Case of $m = 5$

Unfortunately, the Frobenius number $g(3, 4) = 6$ does nothing to achieve closed monochromatic bishops' tours of the $5 \times n$ board. So, once again, we present base cases and a method to extend them to any $5 \times n$ board.

[1] Since this occurs at the right-most end of the $3 \times n$ boards the $2-4$ edge does exist as outlined when creating the closed $3 \times n$ boards.

1	*d*	7	*j*
e	9	*b*	6
8	*a*	10	*c*
i	4	*g*	2
5	*h*	3	*f*

1	*a*	3	*k*	5
b	2	*l*	4	*j*
8	*c*	12	*i*	10
d	7	*f*	11	*h*
6	*e*	9	*g*	13

1	*a*	13	*d*	11	*g*
b	15	*o*	12	*h*	10
14	c	2	*m*	9	*i*
e	4	*l*	6	*j*	8
3	*f*	5	*k*	7	*n*

1	*a*	3	*d*	9	*n*	15
b	2	*q*	10	*o*	14	*m*
4	*c*	8	*p*	16	*l*	13
e	6	*g*	17	*i*	12	*k*
7	*f*	5	*h*	18	*j*	11

Figure 4: Base Cases for $m = 5$

$n \mod 4$	Delete Black	Create Black
0	$2-3$ left $5 \times n, 7-8$ right 5×4	$2-7, 3-8$
1	$3-4$ left $5 \times n, i-h$ right 5×4	$3-i, 4-h$
2	$7-8$ left $5 \times n, 7-8$ right 5×4	$7-7, 8-8$
3	$13-14$ left $5 \times n, i-h$ right 5×4	$13-i, 14-h$
$n \mod 4$	Delete White	Create White
0	$b-c$ left $5 \times n, i-h$ right 5×4	$b-i, c-h$
1	$g-h$ left $5 \times n, 7-8$ right 5×4	$7-g, 8-h$
2	$i-h$ left $5 \times n, i-h$ right 5×4	$h-h, i-i$
3	$j-k$ left $5 \times n, 7-8$ right 5×4	$7-j, 8-k$

7 Summary and Future Work

The above work leads to a very similar looking theorem for bishops as the one proven by Schwenk for knights.

Theorem 2 *An $m \times n$ chessboard with $m \leq n$ has a closed monochromatic bishops' tour unless one of the following three conditions hold:*

(a) $m = 1$;
(b) $m = 2$ and $n \neq 2$;
(c) $m = 3$ and $n = 3$.

Bishops on the $m \times n$ chessboard is not the only setting where a closed monochromatic tour makes sense. Generalizing the square chessboard to a three-dimensional cube also leads naturally to a closed monochromatic tour problem not only for bishops but also for knights. The extension of the bishops' movement into three dimensions is easy to make and clearly still color locked. The extension of a knight's move is not as obvious. One option is to keep the same $1-2$ move of a knight which makes the knight alternate colors on each move. In 2006, Qing and Watkins proved the existence of a closed knight's tour of the six exterior faces of the $i \times j \times k$ rectangular prism in [5] while using the $1-2$ knight. In the same paper, Qing and Watkins provide a $1-2-4$ knight's tour of the cube of side 8. In 2007, DeMaio proved a closed knight's tour in the cube of side n using the $1-2$ knight exists if and only if $n \geq 4$ and even [1]. Why do neither articles use the $1-2-3$ knight? Because the knight's moves in the cube do not alternate color with the $1-2-3$ knight! Just like the bishop, the $1-2-3$ knight is color locked. The next step in this research of monochromatic tours is to determine which cubes admit closed monochromatic tours with the $1-2-3$ knight. As a teaser, I leave you with an example of a closed monochromatic $1-2-3$ knight's tour of the cube of side 6, the smallest cube that admits such a tour. Given the symmetry of the cube only the black tour is given below. Rotate the cube 90 degrees for the white tour.

47		65		6	
	106		74		92
77		99		103	
	54		1		33
24		89		26	
	37		12		45

Level 1

	108		78		41
59		25		67	
	27		46		13
84		93		97	
	75		105		71
2		19		55	

Level 2

16		91		34	
	104		76		101
38		80		44	
	66		60		7
11		50		36	
	95		21		87

Level 3

	73		98		81
53		42		64	
	48		88		5
107		102		90	
	58		100		9
61		23		32	

Level 4

28		68		14	
	85		40		79
94		96		72	
	3		62		51
20		56		18	
	83		70		30

Level 5

	63		52		43
57		17		8	
	15		31		35
69		29		82	
	39		86		22
49		10		4	

Level 6

Figure 5: A Closed Monochromatic $1-2-3$ Knight's Tour in the Cube of Side $n = 6$

References

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