April 10 Math 2306 sec. 51 Spring 2023

A Look Ahead: Solving IVPs

If $f(t)$ is defined on $[0, \infty)$, is differentiable, and has Laplace transform $F(s) = \mathscr{L}\{f(t)\}$, then*

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

Use this result to solve the initial value problem

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

Take $\mathscr{L}$ of the ODE

* See the worksheet 12.

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{4\}$

$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\mathcal{L}\{1\}$

Let $\mathcal{L}\{y(t)\} = Y(s)$

$sY(s) - y(0) + 2Y(s) = \frac{4}{s}$

Isolate $Y(s)$

$sY(s) - 1 + 2Y(s) = \frac{4}{s}$

$sY(s) + 2Y(s) = \frac{4}{s} + 1$

$$(s+2)Y(s) = \frac{4+s}{s}$$

$$Y(s) = \frac{4+s}{s(s+2)}$$

This is the transform of the solution to the IVP

Find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Partial fractions

$$\frac{4+s}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

Clear fractions

$$4+s = A(s+2) + Bs$$

Set $s=0$:

$$4+0 = A(0+2) \Rightarrow A = 2$$

Set $s=-2$ $\quad 4-2 = B(-2) \Rightarrow B = -1$

$$Y(s) = \frac{2}{s} - \frac{1}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{1}{s+2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= 2(1) - e^{-2t}$$

The solution to the IVP is

$$y = 2 - e^{-2t}$$

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

Check the IC: $y(0) = 2 - e^{0} = 2 - 1 = 1$ ✓

Check ODE: $y' = 2e^{-2t}$

$$y' + 2y \stackrel{?}{=} 4$$

$$2e^{-2t} + 2(2 - e^{-2t}) \stackrel{?}{=} 4$$

$$2e^{-2t} + 4 - 2e^{-2t} \stackrel{?}{=} 4$$

$$4 = 4$$ ✓

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

Note that by definition

e^{1t}

$$\mathscr{L}\left\{e^t t^2\right\} = \int_0^\infty e^{-st}e^t t^2\,dt$$

$$= \int_0^\infty e^{-(s-1)t}\, t^2\,dt$$

$$= \int_0^\infty e^{-wt}\, t^2\,dt$$

$$= \frac{2!}{w^3} = \frac{2!}{(s-1)^3}$$

$e^{-st}e^{t} = e^{-st+t}$

$= e^{-(s-1)t}$

Let $w = s-1$

Shift (or translation) in s.

Theorem: Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

We call this a **translation** (or a **shift**) in s theorem.

Example:

Suppose $f(t)$ is a function whose Laplace transform[1]

$$F(s) = \mathscr{L}\{f(t)\} = \frac{1}{\sqrt{s^2+9}}$$

Evaluate

$$\mathscr{L}\left\{e^{-2t}f(t)\right\} = F(s-(-2)) = \frac{1}{\sqrt{(s+2)^2+9}}$$

$$\boxed{\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).}$$

[1] It's not in our table, but this is an actual function known as a *Bessel function*.

Examples: Evaluate

(a) $\mathscr{L}\{t^6 e^{3t}\} = \dfrac{6!}{(s-3)^7}$

$\mathscr{L}\{t^6\} = \dfrac{6!}{s^7} = F(s) \qquad a = 3 \qquad F(s-3)$

(b) $\mathscr{L}\{e^{-t}\cos(t)\} = \dfrac{s+1}{(s+1)^2+1}$

$\mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1^2} \qquad a = -1 \qquad F(s-(-1)) = F(s+1)$

(c) $\mathscr{L}\{e^{-t}\sin(t)\} = \dfrac{1}{(s+1)^2+1}$

$\mathscr{L}\{\sin t\} = \dfrac{1}{s^2+1} = F(s) \qquad F(s+1)$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is irreducible

Complete the square

$s^2+2s+1-1+2 =$

$(s+1)^2+1$

$\dfrac{s}{s^2+2s+2} = \dfrac{s}{(s+1)^2+1}$

Looks like

$\dfrac{s}{s^2+1}$ or $\dfrac{1}{s^2+1}$

we need $s+1$ in place of every s.

$$s = s+1-1$$

$$\frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1}$$

$$= \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$