April 10 Math 2306 sec. 52 Spring 2023

A Look Ahead: Solving IVPs

If $f(t)$ is defined on $[0,\infty)$, is differentiable, and has Laplace transform $F(s) = \mathscr{L}\{f(t)\}$, then*

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

Use this result to solve the initial value problem

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

* See the worksheet 12.

Take $\mathscr{L}$ of the ODE

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{4\}$$

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\,\mathcal{L}\{1\}$$

Let $\mathcal{L}\{y(t)\} = Y(s)$

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s}$$

Plug in $y(0) = 1$ and isolate $Y(s)$

$$sY(s) - 1 + 2Y(s) = \frac{4}{s}$$

$$sY(s) + 2Y(s) = \frac{4}{s} + 1$$

$$(s+2)Y(s) = \frac{4+s}{s}$$

$$Y(s) = \frac{4+s}{s(s+2)}$$ $\Longleftrightarrow$ This is the tranform of the solution to the IVP

Find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

Partial fraction:

$$\frac{4+s}{s(s+2)} = \frac{A}{s} + \frac{B}{s+s}$$

$$4+s = A(s+2) + Bs$$

set $s=0$:

$4+0 = A(2+0) \rightarrow A = 2$

set $s=-2$:

$4-2 = B(-2) \rightarrow B = -1$

$$\Downarrow$$

$$Y(s) = \frac{2}{s} - \frac{1}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= 2\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= 2(1) - e^{-2t}$$

The solution to the IVP is:

$$\boxed{y = 2 - e^{-2t}}$$

$y'(t) + 2y(t) = 4, \quad y(0) = 1$

Check the IC (Initial Condition): $y(0) = 2 - e^{-2(0)} = 2 - 1 = 1$ ✓

Check the ODE: $y' = 2e^{-2t}$

$y' + 2y \stackrel{?}{=} 4$

$2e^{-2t} + 2(2 - e^{-2t}) = 4$

$2e^{-2t} + 4 - 2e^{-2t} = 4$

$4 = 4$ ✓

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

Note that by definition

$$e^{1t} \quad \mathscr{L}\left\{e^t t^2\right\} = \int_0^\infty e^{-st} e^t t^2\,dt$$

$$e^{-st} \cdot e^{t} = e^{t-st} = e^{-(s-1)t}$$

$$= \int_0^\infty e^{-(s-1)t}\, t^2\,dt$$

Let $\omega = s-1$

$$= \int_0^\infty e^{-\omega t}\, t^2\,dt$$

$$= \frac{2!}{\omega^3} = \frac{2!}{(s-1)^3}$$

Shift (or translation) in s.

Theorem: Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

We call this a **translation** (or a **shift**) in s theorem.

Example:

Suppose $f(t)$ is a function whose Laplace transform[1]

$$F(s) = \mathscr{L}\left\{f(t)\right\} = \frac{1}{\sqrt{s^2+9}}$$

Evaluate

$$\mathscr{L}\left\{e^{-2t}f(t)\right\} = \frac{1}{\sqrt{(s+2)^2+9}}$$

[1] It's not in our table, but this is an actual function known as a *Bessel function*.

Examples: Evaluate

(a) $\mathscr{L}\{t^6 e^{3t}\} = \dfrac{6!}{(s-3)^7}$, $\mathscr{L}\{t^6\} = \dfrac{6!}{s^7} = F(s)$

(b) $\mathscr{L}\{e^{-t}\cos(t)\} = \dfrac{s+1}{(s+1)^2+1}$, $\mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1} = F(s)$

$a = -1$ $\quad F(s-(-1)) = F(s+1)$

(c) $\mathscr{L}\{e^{-t}\sin(t)\} = \dfrac{1}{(s+1)^2+1}$, $\mathscr{L}\{\sin t\} = \dfrac{1}{s^2+1}$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is irreducible $\Rightarrow$ Complete the square

$s^2+2s+1-1+2 \Rightarrow (s+1)^2+1$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\}$$

$\dfrac{s}{(s+1)^2+1}$ need $s+1$ where every s is

Note $s = s+1-1$

$$\frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}\{F(s)\}$

$\dfrac{s}{s^2+1^2}$ $\qquad$ $\dfrac{1}{s^2+1^2}$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$