April 10 Math 3260 sec. 51 Spring 2024

Section 5.2: The Characteristic Equation

Recall

- For $n \times n$ matrix A, a nonzero vector $\mathbf{x}$ and scalar λ are an **eigenvector–eigenvalue** pair provided $A\mathbf{x} = \lambda\mathbf{x}$.
- For eigenvalue λ, the subspace $\{\mathbf{x} \mid A\mathbf{x} = \lambda\mathbf{x}\} = \mathrm{Nul}(A - \lambda I)$ is called the **eigenspace** of A corresponding to λ.
- The polynomial equation $\det(A - \lambda I) = 0$ is called the **characteristic equation**, and λ is an eigenvalue if and only if it's a solution to this equation.

Recall

- The eigenvalues of a triangular matrix are the diagonal entries.
- A matrix A is singular if and only if $\lambda = 0$ is an eigenvalue (i.e., it's invertible if and only if $\lambda = 0$ is NOT an eigenvalue).
- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation.
- The **geometric multiplicity** of an eigenvalue is the dimension of its corresponding eigenspace.

Similarity

Definition:

Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

$A = PBP^{-1}$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[a].

[a] **Note:** similarity is NOT related to row equivalence.

Theorem:

If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

Example

Show that $A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ are similar with the matrix P for the similarity transformation given by $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

$B = P^{-1}AP$. Let's find P^{-1}.

$\det(P) = 2(1) - 1(3) = -1$. $P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 6 & -12 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

$-36+42$

$-54+42=-12$

$= B$ so $B = P^{-1}AP$

Example Continued...

Show that the columns of P are eigenvectors of A where

$$A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

$$= [\vec{p}_1 \quad \vec{p}_2]$$

we need to show that $A\vec{p}_1 = \lambda_1 \vec{p}_1$ and

$A\vec{p}_2 = \lambda_2 \vec{p}_2$

$$A\vec{p}_1 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A\vec{p}_2 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Eigenvalues of a real matrix need not be real

Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$.

Solve $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 3 \\ -5 & 2-\lambda \end{bmatrix} = (4-\lambda)(2-\lambda) + 15$$

$$= \lambda^2 - 6\lambda + 8 + 15$$

$$= \lambda^2 - 6\lambda + 23$$

The characteristic equation is

$$\lambda^2 - 6\lambda + 23 = 0$$

Using the quadratic formula

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(23)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-92}}{2}$$

$$= \frac{6 \pm \sqrt{-56}}{2} = \frac{6 \pm 2\sqrt{14}\, i}{2}$$

$$= 3 \pm \sqrt{14}\, i$$

A has no eigenvectors in $\mathbb{R}^2$.

Section 5.3: Diagonalization

Motivational Example:

Determine the eigenvalues of the matrix D^6 (that's D raised to the sixth power), where $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. We can find D^6.

$$D^2 = DD = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & (-1)^2 \end{bmatrix}.$$

$$D^3 = D^2 D = \begin{bmatrix} 2^2 & 0 \\ 0 & (-1)^2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 \\ 0 & (-1)^3 \end{bmatrix}$$

$$\vdots$$

$$D^6 = \begin{bmatrix} 2^6 & 0 \\ 0 & (-1)^6 \end{bmatrix}$$

$$\lambda_1 = 2^6$$

$$\lambda_2 = (-1)^6$$

Recall that a matrix D is diagonal if it is both upper and lower triangular (its only nonzero entries are on the diagonal).

Theorem

If D is a diagonal matrix with diagonal entries d_{ii}, then D^k is diagonal with diagonal entries d_{ii}^k for positive integer k. Moreover, the eigenvalues of D are the diagonal entries.

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \quad \Longrightarrow \quad D^k = \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix}$$

Powers and Similarity

Suppose A and B are similar matrices with similarity transform matrix P—i.e., $B = P^{-1}AP$. Show that

a. A^2 and B^2 are similar with the same P,
b. A^3 and B^3 are similar with the same P.

Given $B = P^{-1}AP$.

$$B^2 = BB$$
$$= (P^{-1}AP)(P^{-1}AP)$$
$$= P^{-1}A\underbrace{(PP^{-1})}_{I}AP$$
$$= P^{-1}AAP \quad = P^{-1}A^2P$$

$$
\begin{aligned}
B^3 &= B^2 B \\
&= (P^{-1} A^2 P)(P^{-1} A P) \\
&= P^{-1} A (P P^{-1}) A P \\
&= P^{-1} A^2 A P \\
&= P^{-1} A^3 P
\end{aligned}
$$

In general, $B^k = P^{-1} A^k P$

for $k \geq 2$.

Diagonalizability

Defintion:

An $n \times n$ matrix A is called **diagonalizable** if it is similar to a diagonal matrix D. That is, provided there exists a nonsingular matrix P such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

Theorem:

The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, the matrix P is the matrix whose columns are the n linearly independent eigenvectors of A.

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

- Find eigenvalues,
- Find eigenvectors,
- If there are 3 lin. ind. evectors,
  form P.

Find e.val.

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)\left[-(5+\lambda)(1-\lambda)+9\right] - 3\left(-3(1-\lambda)+9\right) + 3\left(-9 + 3(+5+\lambda)\right)$$

$$= (1-\lambda)\left(-(-\lambda^2-4\lambda+5)+9\right) - 3\left(3\lambda+6\right) + 3\left(-9+15+3\lambda\right)$$

$$= (1-\lambda)\left(\lambda^2+4\lambda+4\right) \cancel{-3\left(3\lambda+6\right)} \cancel{+3\left(3\lambda+6\right)}$$

$$= (1-\lambda)(\lambda+2)^2$$

Characteristic eqn is $(1-\lambda)(\lambda+2)^2 = 0$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -2$

Find eigenvectors.

For $\lambda_1 = 1$, solve $(A - \lambda_1 I)\vec{x} = \vec{0}$

$$A - 1I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3$

$x_2 = -x_3$.

x_3 - free

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{let } \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

For $\lambda_{2,3} = -2$ $\quad (A + 2I)\vec{x} = \vec{0}$

$$A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2 - x_3$

x_2, x_3 - free

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

let $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Set $P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$\lambda_1 = 1$ $\lambda_{2,3} = -2$

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

$$D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$