April 11 Math 3260 sec. 51 Spring 2022

Section 5.1: Eigenvectors and Eigenvalues

Definition: Let A be an $n \times n$ matrix. A **nonzero** vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Eigenspace

Definition: Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ.**

Remark: The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

Theorems

Theorem: If A is an $n \times n$ triangular matrix, then the eigenvalues of A are its diagonal elements.

Theorem: A square matrix A is invertible if and only if zero is **not** and eigenvalue.

Theorem (adding more to the invertible matrix theorem)

The $n \times n$ matrix A is invertible if and only if[1]

(s) The number 0 is not an eigenvalue of A.

(t) The determinant of A is nonzero.

[1] This is nothing new, we're just adding to the list.

Theorems

Theorem: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, $\lambda_1, \ldots, \lambda_p$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.

Linear Independence

Show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of a matrix A with corresponding eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Consider the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

We'll create 2 equations from this; one by multiplying by A and a second by multiplying by λ_1.

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A\vec{0} = \vec{0}$$

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0}$$

$$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 = \vec{0}$$

$$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_1 \vec{v}_2 = \vec{0}$$

subtract $\quad c_2 (\lambda_2 - \lambda_1) \vec{v}_2 = \vec{0}$

$\vec{v}_2 \neq \vec{0}$ because it's an eigenvector

$\lambda_2 - \lambda_1 \neq 0$. because $\lambda_1 \neq \lambda_2$

Hence $\quad c_2 = 0$.

The original equation becomes

$$c_1 \vec{v}_1 = \vec{0}$$

As an eigenvector, $\vec{v}_1 \neq \vec{0}$, hence $c_1 = 0$

That is, $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ by appealing to the fact that the equation $A\mathbf{x} = \lambda\mathbf{x}$ can be restated as:

using $\vec{x} = I\vec{x}$, $\lambda\vec{x} = \lambda I\vec{x}$. $\Rightarrow A\vec{x} = \lambda I\vec{x}$

Find a nontrivial solution of the homogeneous equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

There is a nontrivial solution if and only if $A - \lambda I$ is singular. This will be true if $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = (2-\lambda)(-6-\lambda) - 3\cdot 3$$

$$= \lambda^2 + 4\lambda - 12 - 9$$

$$= \lambda^2 + 4\lambda - 21$$

we need $\det(A-\lambda I) = 0$. Solve

$$\lambda^2 + 4\lambda - 21 = 0$$

$$(\lambda+7)(\lambda-3) = 0 \implies \lambda = -7 \text{ or } \lambda = 3$$

Let's verify:

Can we find $\vec{x}$ such that $A\vec{x} = -7\vec{x}$

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -7\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{array}{l} x_1 = -\frac{1}{3}x_2 \\ x_2 - \text{free} \end{array}$$

$$\vec{x} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$ so $\lambda_1 = 7$ is an eigenvalue

We can go through the same process with $\lambda_2 = 3$.

Characteristic Equation

Definition: For $n \times n$ matrix A, the expression

$$\det(A - \lambda I)$$

is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition: The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

Theorem: The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Example

Find the characteristic equation for the matrix and identify all of its eigenvalues.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

Set up $A - \lambda I$

$$\det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$$= (5-\lambda)^2(3-\lambda)(1-\lambda)$$

This is $\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75$

The eigenvalues are $5, 3,$ and 1.

Multiplicities

Definition: The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

Example Find the algebraic and geometric multiplicity of the eigenvalue $\lambda = 5$ of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

The characteristic eqn was

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

The algebraic multiplicity of 5 is two.

To find the geometric multiplicity, we find a basis for the eigenspace.

$$A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$A - 5I \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

x_1 - free

$x_2 = 0$

$x_3 = 0$

$x_4 = 0$

A solution $(A - 5I)\vec{x} = \vec{0}$ will look like

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

The geometric multiplicity of the eigenvalue $\lambda = 5$. is one.

Similarity

Definition: Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[2].

Theorem: If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

[2] **Note that similarity is NOT related to being row equivalent.**

If $B = P^{-1}AP$, then $\det(B - \lambda I) = \det(A - \lambda I)$

$$B - \lambda I = P^{-1}AP - \lambda I \qquad I = P^{-1}P$$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}(AP - \lambda P)$$

$$= P^{-1}(A - \lambda I)P$$

Now, take the determinant.

$$\det(B - \lambda I) = \det\left(P^{-1}(A - \lambda I)P\right)$$

$$= \det(P^{-1})\det(A - \lambda I)\det(P)$$

$$= \underbrace{\det(P^{-1})\det(P)}_{1}\det(A-\lambda I)$$

$$= \det(A-\lambda I).$$