April 11 Math 3260 sec. 52 Spring 2022

Section 5.1: Eigenvectors and Eigenvalues

Definition: Let A be an $n \times n$ matrix. A **nonzero** vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Eigenspace

Definition: Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ.**

Remark: The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

Theorems

Theorem: If A is an $n \times n$ triangular matrix, then the eigenvalues of A are its diagonal elements.

Theorem: A square matrix A is invertible if and only if zero is **not** and eigenvalue.

Theorem (adding more to the invertible matrix theorem)

The $n \times n$ matrix A is invertible if and only if[1]

(s) The number 0 is not an eigenvalue of A.

(t) The determinant of A is nonzero.

[1] This is nothing new, we're just adding to the list.

Theorems

Theorem: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, $\lambda_1, \ldots, \lambda_p$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.

Linear Independence

Show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of a matrix A with corresponding eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

we can consider the homogeneous vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

we'll create two equations from this; one by multiplying by the matrix A, and the 2nd by multiplying by the number λ_2.

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A\vec{0} = \vec{0}$$

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0}$$

$$c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 = \vec{0}$$

$$c_1 \lambda_2 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 = \vec{0}$$

subtract $\quad c_1(\lambda_1 - \lambda_2)\vec{v}_1 = \vec{0}$

$\vec{v}_1 \neq \vec{0}$ because $\vec{v}_1$ is an eigenvector

$\lambda_1 - \lambda_2 \neq 0$ because they are different eigenvalues

Hence $c_1 = 0$.

The equation becomes

$$c_2 \vec{v}_2 = \vec{0}$$

Since $\vec{v}_2$ is an eigenvector, $\vec{v}_2 \neq \vec{0}$.

So $c_2 = 0$.

That is $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ only if

$c_1 = c_2 = 0$. So $\{\vec{v}_1, \vec{v}_2\}$ is

linearly independent.

Section 5.2: The Characteristic Equation

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ by appealing to the fact that the equation $A\mathbf{x} = \lambda\mathbf{x}$ can be restated as: use $\vec{x} = I\vec{x}$

$A\vec{x} = \lambda\vec{x} \Rightarrow A\vec{x} = \lambda I\vec{x} \Rightarrow A\vec{x} - \lambda I\vec{x} = \vec{0} \Rightarrow (A - \lambda I)\vec{x} = \vec{0}$

Find a nontrivial solution of the homogeneous equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

This requires the matrix $A - \lambda I$ to be singular.

This requires its determinant to be zero.

we need to solve the equation

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - 3 \cdot 3$$
$$= \lambda^2 + 4\lambda - 12 - 9$$
$$= \lambda^2 + 4\lambda - 21$$

$$\det(A - \lambda I) = 0 \implies \lambda^2 + 4\lambda - 21 = 0$$
$$(\lambda + 7)(\lambda - 3) = 0 \implies \lambda = -7 \text{ or } \lambda = 3$$

The eigenvalues are $\lambda_1 = -7$, $\lambda_2 = 3$.

Let's Check $\lambda_2 = 3$. Find $\vec{x}$ such that

$$A\vec{x} = 3\vec{x}$$

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using an augmented matrix

$$\begin{bmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = 3x_2 \\ x_2 - \text{free}. \end{array}$$

So we get eigenvectors

$$\vec{x} = x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Characteristic Equation

Definition: For $n \times n$ matrix A, the expression

$$\det(A - \lambda I)$$

is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition: The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

Theorem: The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Example

Find the characteristic equation for the matrix and identify all of its eigenvalues.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

Solve $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$$= (5-\lambda)^2(3-\lambda)(1-\lambda)$$

$$= \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75$$

The eigen values are $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 1$.

Multiplicities

Definition: The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

Example Find the algebraic and geometric multiplicity of the eigenvalue $\lambda = 5$ of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

The characteristic polynomial is

$$(5-\lambda)^2(3-\lambda)(1-\lambda)$$

The algebraic multiplicity of $\lambda = 5$ is two.

To find the geometric multiplicity, we can find a basis for the eigenspace.

$$A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 is free

$x_2 = 0$

$x_3 = 0$

$x_4 = 0$

The eigenvectors look like

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A basis for the eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The geometric multiplicity of $\lambda = 5$ is one.

Similarity

Definition: Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[2].

Theorem: If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

[2] **Note that similarity is NOT related to being row equivalent.**

If $B = P^{-1}AP$, then $\det(B - \lambda I) = \det(A - \lambda I)$

$$B - \lambda I = P^{-1}AP - \lambda I \qquad I = P^{-1}P$$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}(AP - \lambda P)$$

$$= P^{-1}(A - \lambda I)P$$

Take the determinant of both sides

$$\det(B - \lambda I) = \det\left(P^{-1}(A - \lambda I)P\right)$$

$$= \det(P^{-1})\det(A-\lambda I)\det(P)$$

$$= \underbrace{\det(P^{-1})\det(P)}_{1}\det(A-\lambda I)$$

$$\Rightarrow \quad \det(B-\lambda I) = \det(A-\lambda I)$$