April 12 Math 2306 sec. 51 Spring 2023

Section 15: Shift Theorems

We saw the first of two shift[1] theorems; shift in s.

Theorem: Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

[1] or *translation*

Example

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-3)^8}\right\} = e^{3t}\mathcal{L}^{-1}\left\{\dfrac{1}{s^8}\right\}$

$$= e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{7!}\,\frac{7!}{s^8}\right\}$$

$$\frac{1}{s^8} = F(s)$$

$$= \frac{1}{7!}e^{3t}\mathcal{L}^{-1}\left\{\frac{7!}{s^8}\right\}$$

$$= \frac{1}{7!}e^{3t}t^7$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Partial fractions

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$ Clear fractions

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$= (A+B)s^2 + (-2A-B+C)s + A$$

Match like terms

$$
\begin{aligned}
A \qquad\qquad &= 1 \qquad \Longrightarrow \quad A = 1 \\
-2A - B + C &= 3 \\
A + B \qquad &= -1 \qquad \Rightarrow \quad B = -1 - A = -2
\end{aligned}
$$

$$C = 3 + 2A + B = 3 + 2 - 2 = 3$$

$$\frac{-s^2 + 3s + 1}{s(s-1)^2} = \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{-s^2+3s+1}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= 1 - 2e^{t} + 3e^{t}\,t$$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^{1t}\,\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t\, t$$

$$F(s) = \frac{1}{s^2} = \frac{1!}{s^2}$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Heaviside Step

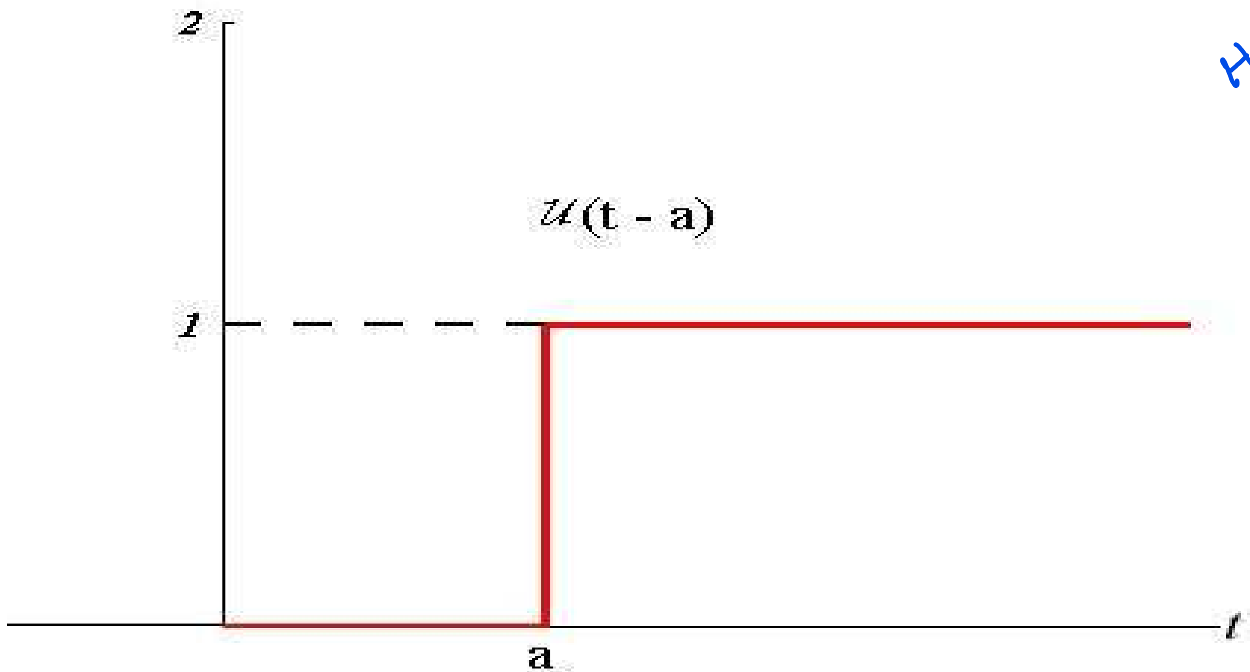

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$a > 0$

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$

$$= g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Consider $0 \le t < a$, then $\mathcal{U}(t-a) = 0$

$g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$

$= g(t) - g(t)(0) + h(t)(0) = g(t) = f(t)$ as expected

$$\begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Consider $t \geq a$, then $\mathcal{U}(t-a) = 1$.

$g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$

$= g(t) - g(t)(1) + h(t)(1) = h(t) = f(t)$

again, as expected

Example $f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$

Rewrite the function f in terms of the unit step function.

$$f(t) = e^t - e^t u(t-2) + t^2 u(t-2) - t^2 u(t-5) + 2t\, u(t-5)$$

$$u(t-0) = 1$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases} .$$

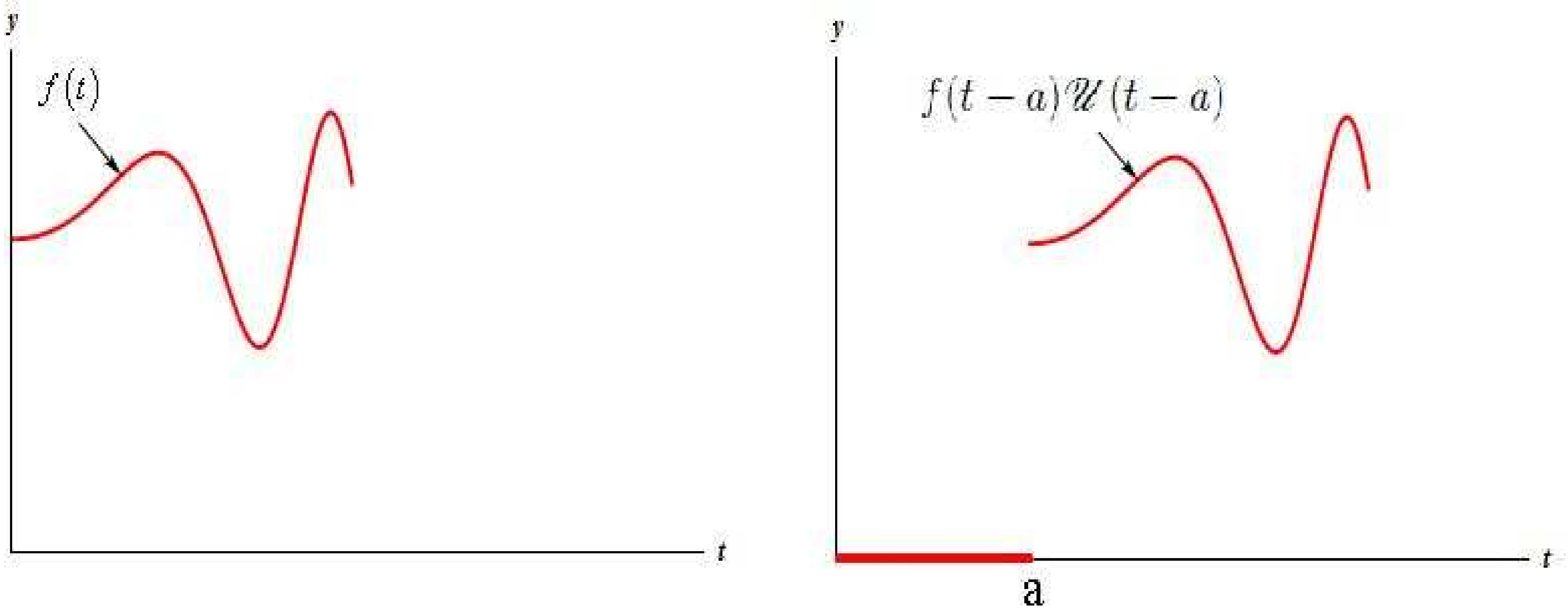

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Here, $f(t) = \underline{1}$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \frac{1}{-s} e^{-st}\Big|_a^\infty$$

convergence requires $s > 0$

$$= \frac{-1}{s}\left(0 - e^{-as}\right)$$

$$= \frac{e^{-as}}{s}$$

Note, this is $e^{-as}\mathcal{L}\{1\}$

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

A special case is $f(t) = 1$. We just found
$\mathscr{L}\{\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{1\} = \dfrac{e^{-as}}{s}$.

We can state this in terms of the inverse transform as

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + \mathcal{U}(t-1)\left(-1+t\right)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathcal{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{\underbrace{(t-1)}_{t}\mathcal{U}(t-1)\}$$

$$\mathscr{L}\{t\} = \frac{1}{s^2}$$

$$= \frac{1}{s} + e^{-1s}\frac{1}{s^2}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}$$