April 12 Math 2306 sec. 52 Spring 2023

Section 15: Shift Theorems

We saw the first of two shift[1] theorems; shift in s.

Theorem: Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

[1] or *translation*

Example

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-3)^8}\right\} = e^{3t}\mathscr{L}^{-1}\left\{\dfrac{1}{s^8}\right\}$

$= e^{3t}\mathscr{L}^{-1}\left\{\dfrac{1}{7!}\dfrac{7!}{s^8}\right\}$

$= \dfrac{1}{7!}e^{3t}\mathscr{L}^{-1}\left\{\dfrac{7!}{s^8}\right\}$

$= \dfrac{1}{7!}e^{3t}t^7$

$\dfrac{1}{s^8} = F(s)$

$\dfrac{1}{s^8} = \dfrac{1}{7!}\dfrac{7!}{s^8}$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

· Partial fractions

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$= (A+B)s^2 + (-2A-B+C)s + A$$

Match like terms

$$A = 1 \quad \Rightarrow A = 1$$

$$-2A - B + C = 3$$

$$A + B = -1 \quad \Rightarrow B = -1 - A = -1 - 1 = -2$$

$$C = 3 + 2A + B = 3 + 2(1) - 2 = 3$$

$$\frac{-s^2 + 3s + 1}{s(s-1)^2} = \frac{1}{s} + \frac{-2}{s-1} + \frac{3}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{-s^2+3s+1}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= 1 - 2e^{1t} + 3te^{t}$$

$$= 1 - 2e^{t} + 3te^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^{1t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{t}\,\mathcal{L}^{-1}\left\{\frac{1!}{s^2}\right\} = e^{t}\,t$$

$$\frac{1}{s^2} = F(s)$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Heaviside
Step

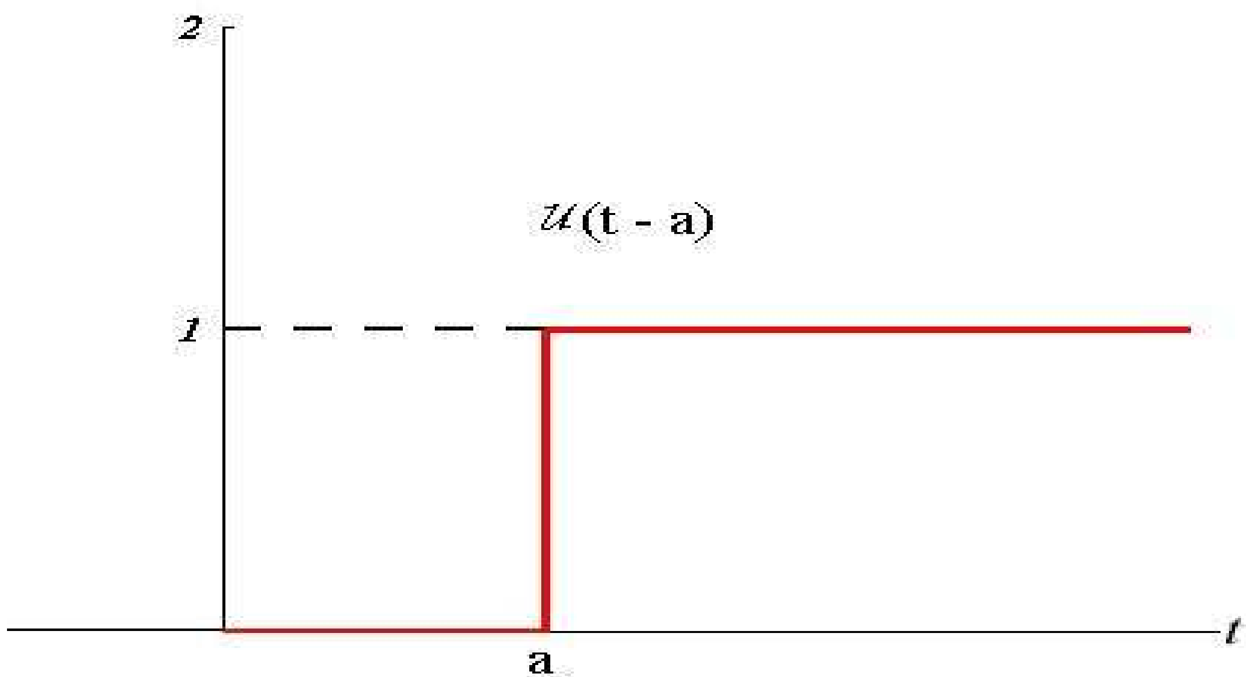

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$a > 0$

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$

$$= g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \geq a \end{cases}$$

Consider $0 \le t < a$, then $\mathscr{U}(t-a) = 0$

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)\cdot(0) + h(t)\cdot(0)$$

$$= g(t)$$

$= f(t)$ as expected

$$\begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Consider $t \geqslant a$, then $\mathcal{U}(t-a) = 1$

$g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a) =$

$g(t) - g(t)\cdot(1) + h(t)\cdot(1) = h(t)$

$= f(t)$ again as expected

Example $f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$

Rewrite the function f in terms of the unit step function.

$$f(t) = e^t - e^t \mathcal{U}(t-2) + t^2 \mathcal{U}(t-2) - t^2 \mathcal{U}(t-5) + 2t\, \mathcal{U}(t-5)$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases} .$$

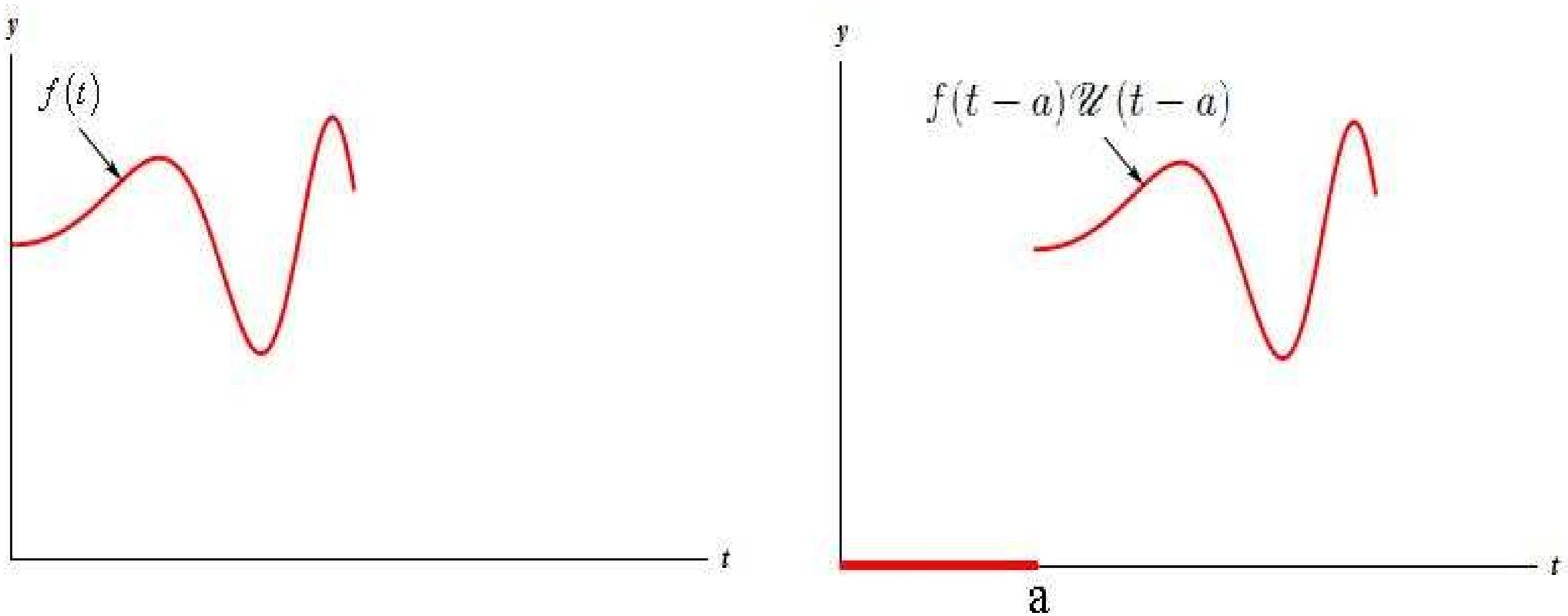

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

$a > 0$

$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

$f(t) = 1$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

convergence requires $s > 0$

$$= \frac{1}{-s} e^{-st}\Big|_a^\infty$$

$$= \frac{-1}{s}\left(0 - e^{-as}\right) = \frac{e^{-as}}{s}$$

Note that this is $e^{-as}\mathcal{L}\{1\}$

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

A special case is $f(t) = 1$. We just found
$\mathscr{L}\{\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{1\} = \dfrac{e^{-as}}{s}$.

We can state this in terms of the inverse transform as

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + \mathcal{U}(t-1)\left(-1 + t\right)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{\underbrace{(t-1)}_{t}\mathscr{U}(t-1)\}$$

$$\mathscr{L}\{t\} = \frac{1!}{s^2}$$

$$= \frac{1}{s} + e^{-1s}\mathscr{L}\{t\}$$

$$= \frac{1}{s} + e^{-s}\left(\frac{1}{s^2}\right) = \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$$\mathscr{L}\{h(t-a)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{h(t)\}$$

$h(t) = ?$ If $h(t-1) = t-1$

Alternative Form for Translation in t

It is often the case that we wish to take the transform of a product of the form

$$g(t)\mathscr{U}(t-a)$$

in which the function g is not translated.

The main theorem statement

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

can be restated as

$$\boxed{\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.}$$

This is based on the observation that

$$g(t) = g((t+a)-a).$$