April 13 Math 3260 sec. 52 Spring 2022

Section 5.2: The Characteristic Equation

Definition: For $n \times n$ matrix A, the expression

$$\det(A - \lambda I)$$

is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition: The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

Eigenvalues & Multiplicities

Theorem: The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Definition: The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

The two multiplicities can be different. The geometric multiplicity is always $\leq$ the algebraic multiplicity.

Similarity

Definition: Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation.**

Theorem: If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

Example

Show that $A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ are similar with the matrix P for the similarity transformation given by $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

We need to show that $B = P^{-1}AP$.

Find P^{-1}.

$$\det(P) = 2\cdot 1 - 1\cdot 3 = -1$$

$$P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

Example Continued...

Show that the columns of P are eigenvectors of A where

$$A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}. = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix}$$

We need to show that $A\vec{p}_1 = \lambda_1 \vec{p}_1$ and $A\vec{p}_2 = \lambda_2 \vec{p}_2$ for some numbers λ_1 and λ_2.

$$A\vec{p}_1 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A\vec{p}_2 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We see that $\vec{p}_1$ is an eigenvector w/ eigenvalue 3 and $\vec{p}_2$ is an eigenvector w/ eigenvalue -4.

Eigenvalues of a real matrix need not be real

Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$.

$$\det(A-\lambda I) = \det \begin{bmatrix} 4-\lambda & 3 \\ -5 & 2-\lambda \end{bmatrix} = (4-\lambda)(2-\lambda) - (-5)\cdot 3$$

$$= \lambda^2 - 6\lambda + 23$$

The characteristic equation is

$$\lambda^2 - 6\lambda + 23 = 0 \implies \lambda^2 - 6\lambda + 9 = -23 + 9$$

$$(\lambda - 3)^2 = -14 \implies \lambda = 3 \pm \sqrt{14}\, i$$