April 14 Math 2306 sec. 51 Spring 2023

Section 15: Shift Theorems

We have two major shift (a.k.a. translation) theorems.

Theorem: Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

Theorem: If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

Recall the Unit Step Function

$$\text{For } a > 0, \quad \mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

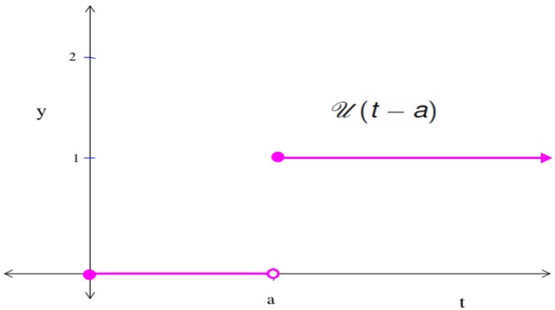

Figure: The function $\mathscr{U}(t-0) = \mathscr{U}(t) = 1$ for $t > 0$. Here, we are taking $\mathscr{U}(0) = 1$, but depending on the context, it might be taken as 0 or $\frac{1}{2}$.

Alternative Form for Translation in t

It is often the case that we wish to take the transform of a product of the form

$$g(t)\mathscr{U}(t-a)$$

in which the function g is not translated.

The main theorem statement

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

can be restated as

$$\boxed{\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.}$$

This is based on the observation that

$$g(t) = g((t+a)-a).$$

Example

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

Example: Find $\mathscr{L}\left\{\cos t\, \mathscr{U}\left(t-\frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}s}\mathscr{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$

$a = \frac{\pi}{2}$

$g(t) = \cos t$

$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\}$

$= -e^{-\frac{\pi}{2}s}\frac{1}{s^2+1^2} = \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}$

$\cos\left(t+\frac{\pi}{2}\right) = \cos t \underbrace{\cos\frac{\pi}{2}}_{=0} - \sin t \underbrace{\sin\frac{\pi}{2}}_{=1}$

Example

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

Need $f(t) = \mathscr{L}^{-1}\{F(s)\}$

Here $F(s) = \dfrac{1}{s(s+1)}$

Partial fractions

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$1 = A(s+1) + Bs$$

$s = 0 \qquad 1 = A \qquad \Rightarrow \quad \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$s = -1 \qquad 1 = -B$

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1}{s+1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{s+1} \right\}$$

$$= 1 - e^{-t}$$

$$f(t) = 1 - e^{-t}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \left(1 - e^{-(t-2)}\right)\mathscr{U}(t-2)$$

$a = 2$