April 12 Math 3260 sec. 51 Spring 2024

Section 6.1: Inner Product, Length, and Orthogonality

Definition

For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ we define the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ (also called the **dot product**) by the **matrix product**

$$\mathbf{u}^T\mathbf{v} = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Remark: Note that this product produces a scalar. It is sometimes called a *scalar product*. There are several notations for this:

$$\mathbf{u}^T\mathbf{v} = \mathbf{u}\cdot\mathbf{v} = \langle\mathbf{u},\mathbf{v}\rangle.$$

Inner Product Properties

The dot product is an example of a class of functions that take two elements of a Real vector space and assign a scalar value. An **Inner Product** must satisfy the following properties.

Inner Product Properties

For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and any scalar c

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (commutitivity)
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ (distributive property)
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ (factoring)
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$ (positive definiteness)

The Norm

That last property, being positive definite, allows us to define a **norm**.

Definition

The **norm** of the vector $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is the nonnegative number

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Remark: This is sometimes called the 2-norm, and might be written like $\|\mathbf{v}\|_2$. It corresponds to what we traditionally think of as *length* of a vector as a directed line segment.

Remark: This norm is often referred to as the **magnitude** of a vector.

Theorem

Theorem

For vector **v** in $\mathbb{R}^n$ and scalar c

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Note $\|c\vec{v}\|^2 = (c\vec{v})\cdot(c\vec{v}) = c(c)\,\vec{v}\cdot\vec{v}$

$$= c^2\|\vec{v}\|^2$$

Taking square roots

$$\|c\vec{v}\| = \sqrt{c^2\|\vec{v}\|^2} = |c|\,\|\vec{v}\|$$

Unit Vectors & Normalizing

Definition

A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $\|\mathbf{u}\| = 1$ is called a **unit vector.**

Example: Show that $\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ is a unit vector.

$$\|\vec{x}\|^2 = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2$$

$$= \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

$$\Rightarrow \|\vec{x}\| = \sqrt{1} = 1$$

Unit Vectors & Normalizing

Remark

Given any nonzero vector **v** in $\mathbb{R}^n$, we can find a unit vector in the direction of **v** by dividing **v** by its norm. This is called **normalizing** the vector.

Show that if **v** is nonzero, then $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

$$\|\vec{u}\| = \left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \left| \frac{1}{\|\vec{v}\|} \right| \|\vec{v}\|$$

$$= \frac{1}{\|\vec{v}\|} \|\vec{v}\| = 1$$

Distance in $\mathbb{R}^n$

Definition:

For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v** is denoted by

$$\text{dist}(\mathbf{u}, \mathbf{v}),$$

and is defined by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Remark: This is the same as the traditional formula for distance used in $\mathbb{R}^2$ between points (x_0, y_0) and (x_1, y_1),

$$d = \sqrt{(y_1 - y_0)^2 + (x_1 - x_0)^2}.$$

Example

Find the distance between the vectors $\mathbf{u} = (4, 0, -1, 1)$ and $\mathbf{v} = (0, 0, 2, 7)$ in $\mathbb{R}^4$.

$\text{dist}(\vec{u}, \vec{v}) = 7.81 = \sqrt{61}$

$\vec{u} - \vec{v} = (4, 0, -3, -6)$

$\text{dist}(\vec{u}, \vec{v}) = \sqrt{4^2 + 0^2 (-3)^2 + (-6)^2}$

Orthogonality

Definition:

Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

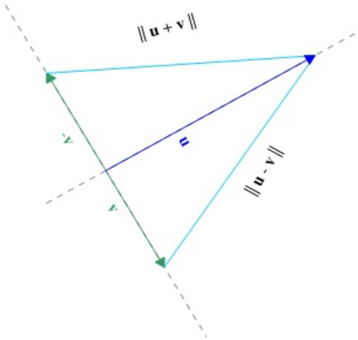

Figure: Note that two vectors are perpendicular if $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$

Orthogonal and Perpendicular

Show that $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Note that

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$$

$$= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v}$$

Similarly

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u} \cdot \vec{v}$$

From this, we see that

$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$ i.e., $\|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\|$

if $\vec{u}\cdot\vec{v} = 0$

And if $\vec{u}\cdot\vec{v} = 0$, then $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$

making $\|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\|$

The Pythagorean Theorem

Theorem:

Two vectors **u** and **v** are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

This follows immediately from the observation that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}.$$

The two vectors are defined as being orthogonal precisely when $\mathbf{u}\cdot\mathbf{v} = 0$.

Orthogonal Complement

Definition:

Let W be a subspace of $\mathbb{R}^n$. A vector $\mathbf{z}$ in $\mathbb{R}^n$ is said to be **orthogonal to** W if $\mathbf{z}$ is orthogonal to every vector in W. That is, if

$$\mathbf{z} \cdot \mathbf{w} = 0 \quad \text{for every} \quad \mathbf{w} \in W.$$

Definition:

Given a subspace W of $\mathbb{R}^n$, the set of all vectors orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^\perp$ (read as “W perp”).

$$W^\perp = \left\{ \mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x} \cdot \mathbf{w} = 0 \quad \text{for every} \quad \mathbf{w} \in W \right\}$$

Theorem:

Theorem:

If W is a subspace of $\mathbb{R}^n$, then $W^\perp$ is a subspace of $\mathbb{R}^n$.

This is readily proved by appealing to the properties of the inner product. In particular

$$\begin{array}{ll} (1) & \mathbf{0}\cdot\mathbf{w}=0 \quad \text{for any vector} \quad \mathbf{w} \\ (2) & (\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w} \quad \text{and} \\ (3) & (c\mathbf{u})\cdot\mathbf{w}=c\mathbf{u}\cdot\mathbf{w}. \end{array}$$

(1) The zero vector is in $W^\perp$.

(2) If $\mathbf{u}$ and $\mathbf{v}$ are in $W^\perp$, then so is $\mathbf{u}+\mathbf{v}$.

(3) If $\mathbf{u}$ is in $W^\perp$, then so is $c\mathbf{u}$ for any scalar c.

Example

$\vec{e}_1$ $\vec{e}_3$

Let W =Span$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Then $W^\perp$ =Span$\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

A vector in W has the form

$$\mathbf{w} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}x\\0\\z\end{bmatrix}.$$

A vector in $\mathbf{v}$ in $W^\perp$ has the form

$$\mathbf{v} = y\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\y\\0\end{bmatrix}.$$

Note that

$$\mathbf{w}\cdot\mathbf{v} = x(0) + 0(y) + z(0) = 0.$$

W is the xz-plane and $W^\perp$ is the y-axis in $\mathbb{R}^3$.

Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix}$. Show that if $\mathbf{x}$ is in Nul(A), then $\mathbf{x}$ is in $[\text{Row}(A)]^\perp$.

we need to show that if $\vec{x}$ is in Nul(A), then $\vec{x} \cdot \vec{u} = 0$ for all $\vec{u}$ in Row(A).

Let's characterize Nul(A) and Row(A).

Note, $\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \right\}$. For the null space, use $[A \ \vec{0}]$.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ -2 & 0 & 4 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4/3 & 0 \end{bmatrix}$$

$x_1 = 2x_3$
$x_2 = -4/3\, x_3$
x_3 - free

$$\vec{x} = x_3 \begin{bmatrix} 2 \\ -4/3 \\ 1 \end{bmatrix} = \frac{x_3}{3} \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$$

So $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix} \right\}$.

Let $\vec{x}$ be in Nul(A), $\vec{x} = t \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$. Let $\vec{u}$ be in Row(A), so $\vec{u} = c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$.

$$\vec{x} \cdot \vec{u} = \left(t \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix} \right) \cdot \left(c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \right)$$

$$= tc_1(6-12+6) + tc_2(-12+0+12)$$

$$= tc_1(0) + tc_2(0) = 0$$

So $\vec{x}$ is in $[\mathrm{Row}(A)]^\perp$

$$A\vec{x} = \begin{bmatrix} \text{---} \end{bmatrix}\begin{bmatrix} | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The Fundamental Subspaces of a Matrix

Theorem:

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A. That is

$$[\mathrm{Row}(A)]^\perp = \mathrm{Nul}(A).$$

The orthogonal complement of the column space of A is the null space of A^T—i.e.

$$[\mathrm{Col}(A)]^\perp = \mathrm{Nul}(A^T).$$

Example: Find an orthogonal complement.

Let $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ 1 \\ -10 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ -1 \\ 13 \end{bmatrix} \right\}$. Find a basis for $W^\perp$.

We can form a matrix A having W as its Row space.

Let $A = \begin{bmatrix} 2 & 4 & 1 & -10 \\ -3 & -6 & -1 & 13 \end{bmatrix}$

$W^\perp = \text{Nul}(A)$

$$\text{rref } A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$$x_1 = -2x_2 + 3x_4$$

$$x_3 = 4x_4$$

x_2, x_4 - free

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$$W^\perp = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}.$$