April 15 Math 3260 sec. 52 Spring 2022

Section 5.3: Diagonalization

Motivating Example:
Determine the eigenvalues of the matrix D^3 (that's D cubed), where
$D = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$.

$$D^2 = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & -64 \end{bmatrix}$$

$$= \begin{bmatrix} 3^3 & 0 \\ 0 & (-4)^3 \end{bmatrix}$$

The eigenvalues of D^3 are 27 and -64.

Diagonal Matrices

Recall: A matrix D is diagonal if it is both upper and lower triangular (its only nonzero entries are on the diagonal).

Note: If D is diagonal with diagonal entries d_{ii}, then D^k is diagonal with diagonal entries d_{ii}^k for positive integer k. Moreover, the eigenvalues of D are the diagonal entries.

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \quad \Longrightarrow \quad D^k = \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix}$$

Powers and Similarity

Suppose A and B are similar matrices with similarity transform matrix P. Show that

a. A^2 and B^2 are similar with the same P,

b. A^3 and B^3 are similar with the same P.

A and B are similar means

$$B = P^{-1}AP$$

$$B^2 = (P^{-1}AP)^2$$

$$= (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A\underbrace{PP^{-1}}_{I}AP$$

$$= P^{-1}AAP = P^{-1}A^2P$$

$$\Rightarrow B^2 = P^{-1}A^2P$$

$$B^3 = B^2B = (P^{-1}A^2\underbrace{P)(P^{-1}}_{I}AP)$$

$$= P^{-1}A^2AP$$

$$= P^{-1}A^3P.$$

By induction $B^k = P^{-1}A^kP$ for all positive integers k.

Diagonalizability

Defintion: An $n \times n$ matrix A is called **diagonalizable** if it is similar to a diagonal matrix D. That is, provided there exists a nonsingular matrix P such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

Theorem: The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, the matrix P is the matrix whose columns are the n linearly independent eigenvectors of A.

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Find the eigenvalues.

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -(5+\lambda) & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -(5+\lambda) \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)\left[-(5+\lambda)(1-\lambda)+9\right] - 3\left[-3(1-\lambda)+9\right] + 3\left[-9+3(5+\lambda)\right]$$

$$= (1-\lambda)\left[\lambda^2+4\lambda+4\right] \underbrace{-3\left[3\lambda+6\right]+3\left[3\lambda+6\right]}_{=0}$$

$$= (1-\lambda)(\lambda+2)^2$$

The characteristic equation is

$$(1-\lambda)(\lambda+2)^2=0 \Rightarrow \lambda_1=1 \text{ and } \lambda_2=-2.$$

Find bases for the eigenspaces.

For $\lambda_1=1$

$$A-1I=\begin{bmatrix} 0 & 3 & 3\\ -3 & -6 & -3\\ 3 & 3 & 0\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$x_1 = x_3$
$x_2 = -x_3$
x_3 free

The eigenvectors are $\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -2$

$$A - (-2)I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2 - x_3$

x_2, x_3 are free

Eigenvectors have the form

$$\vec{X} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $\vec{V}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{V}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. We have

3 linearly independent eigenvectors $\Rightarrow$

A is diagonalizable.

$$\vec{V}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{V}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{V}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $P = [\vec{V}_1 \ \vec{V}_2 \ \vec{V}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

$$D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$