April 17 Math 2306 sec. 52 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform[1], $\mathscr{L}\{f(t)\} = F(s)$, and that f is differentiable on $[0, \infty)$. Obtain an expression for the Laplace tranform of $f'(t)$ using integration by parts to get

$$\begin{aligned}
\mathscr{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)\, dt \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)\, dt \\
&= sF(s) - f(0).
\end{aligned}$$

[1] Assume f is of exponential order c for some c.

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(sF(s) - f(0)\right) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\{y(t)\} = Y(s),$$

then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0),\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0),\\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0),\\
&\vdots \qquad \vdots\\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Laplace Transforms and IVPs

For constants a, b, and c, take the Laplace transform of both sides of the equation and isolate $\mathscr{L}\{y(t)\} = Y(s)$.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $\mathscr{L}\{g(t)\} = G(s)$.

Take $\mathscr{L}$ of the ODE

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{g(t)\}$$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = G(s)$$

$$a\left(s^2Y(s)-sy(0)-y'(0)\right)+b\left(sY(s)-y(0)\right)+cY(s)=G(s)$$

$$y(0)=y_0,\quad y'(0)=y_1$$

$$as^2Y(s)-ay_0s-ay_1+bsY(s)-by_0+cY(s)=G(s)$$

$$(as^2+bs+c)Y(s)-ay_0s-ay_1-by_0=G(s)$$

$$(as^2+bs+c)Y(s)=ay_0s+ay_1+by_0+G(s)$$

$$ay''+by'+cy=g(t),$$

Note: The coefficient of Y is the

Characteristic polynomial.

$$\Psi(s) = \frac{a y_0 s + a y_1 + b y_0}{a s^2 + bs + c} + \frac{G(s)}{a s^2 + bs + c}$$

The solution to the IVP

is $$y(t) = \mathcal{L}^{-1}\{\Psi(s)\}.$$

Solving IVPs

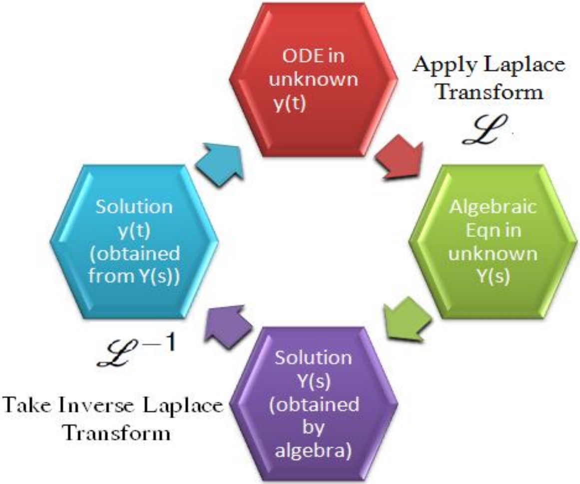

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**

Solve the IVP using the Laplace Transform

$$y''+7y'+12y = e^{-t} \quad y(0) = 2, \quad y'(0) = -6$$

Let $\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{y'' + 7y' + 12y\} = \mathcal{L}\{e^{-t}\}$$

$$\mathcal{L}\{y''\} + 7\mathcal{L}\{y'\} + 12\mathcal{L}\{y\} = \frac{1}{s+1}$$

$$s^2Y(s) - sy(0) - y'(0) + 7\left(sY(s) - y(0)\right) + 12Y(s) = \frac{1}{s+1}$$

$$y(0) = 2, \quad y'(0) = -6$$

$$s^2Y(s) - 2s + 6 + 7\left(sY(s) - 2\right) + 12Y(s) = \frac{1}{s+1}$$

$$(s^2+7s+12)Y(s) - 2s + 6 - 14 = \frac{1}{s+1}$$

$$(s^2+7s+12)Y(s) = \frac{1}{s+1} + 2s + 8$$

$$Y(s) = \frac{1}{(s+1)(s^2+7s+12)} + \frac{2(s+4)}{s^2+7s+12}$$

$$Y(s) = \frac{1}{(s+1)(s+3)(s+4)} + \frac{2(s+4)}{(s+3)(s+4)}$$

$$\Psi(s) = \frac{1}{(s+1)(s+3)(s+4)} + \frac{2}{s+3}.$$

Partial fractions

$$\frac{1}{(s+1)(s+3)(s+4)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$1 = A(s+3)(s+4) + B(s+1)(s+4) + C(s+1)(s+3)$$

set $s=-1$ $\quad 1 = A(2)(3) \implies A = \frac{1}{6}$

$s=-3$ $\quad 1 = B(-2)(1) \implies B = \frac{-1}{2}$

$s=-4$ $\quad 1 = C(-3)(-1) \implies C = \frac{1}{3}$

$$Y(s) = \frac{\frac{1}{6}}{s+1} - \frac{\frac{1}{2}}{s+3} + \frac{\frac{1}{3}}{s+4} + \frac{2}{s+3}$$

$$Y(s) = \frac{\frac{1}{6}}{s+1} + \frac{\frac{3}{2}}{s+3} + \frac{\frac{1}{3}}{s+4}$$

The solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\boxed{y(t) = \frac{1}{6}e^{-t} + \frac{3}{2}e^{-3t} + \frac{1}{3}e^{-4t}}$$

$$y(0) = \frac{1}{6} + \frac{3}{2} + \frac{1}{3} = \frac{1}{6} + \frac{9}{6} + \frac{2}{6} = \frac{12}{6} \checkmark$$

Unit Impulse

Consider the piecewise constant, rectangular function

$$R_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases}$$

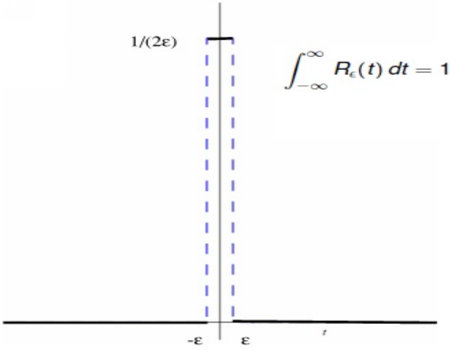

Figure: For every $\epsilon > 0$, the integral of R_ϵ over the real line is 1.

Unit Impulse

We can plot R_ϵ for various values of ϵ and see that as ϵ gets smaller, the rectangle gets narrow and tall. But the area of the rectangle is kept constant at 1.

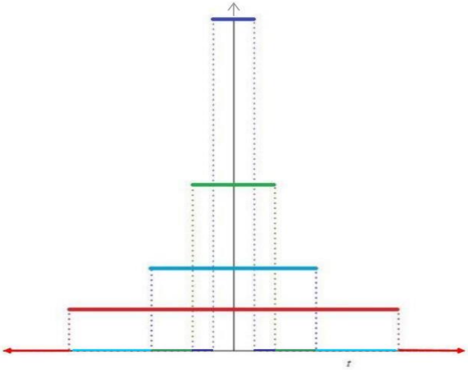

Figure: $R_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases}$

Unit Impulse

The Dirac delta *function*, denoted by $\delta(\cdot)$, models a strong instantaneous force. One way to define this function is as the limit

$$\delta(t) = \lim_{\epsilon \to 0} R_\epsilon(t).$$

This is not a function in the usual sense, but it has several properties.

- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$ for any real number a.
- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)f(t)\,dt = f(a)$ if a is in the domain of the function f.
- $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for any constant $a \geq 0$.

Remark: This is an example of what is called a *generalized function*, *generalized functional*, or *distribution*. In this context, it can be thought of as the derivative of the Heaviside step function. That is, for any $a \geq 0$

$$\frac{d}{dt}\mathscr{U}(t-a) = \delta(t-a).$$

Solve the IVP using the Laplace Transform

A 1 kg mass is suspended from a spring with spring constant 10 N/m. A damper induces damping of 6 N per m/sec of velocity. The object starts from rest from a position 10 cm above equilibrium. At time $t = 1$ second, a unit impulse force is applied to the object. Determine the object's position for $t > 0$.

The corresponding IVP for the situation described is

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

$$mx'' + \beta x' + kx = f(t)$$