April 18 Math 3260 sec. 52 Spring 2022

Section 5.3: Diagonalization

Defintion: An $n \times n$ matrix A is called **diagonalizable** if it is similar to a diagonal matrix D. That is, provided there exists a nonsingular matrix P such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

Theorem: The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, the matrix P is the matrix whose columns are the n linearly independent eigenvectors of A.

Remark: The diagonal matrix D will have the eigenvalues of A on its main diagonal. The order will correspond to the order in which the eigenvectors are used to construct the matrix P.

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

The eigenvalues were $\lambda_1 = 1$ and $\lambda_2 = -2$. We found three linearly independent eigenvectors, so A is diagonalizable. We found a P and D

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. (With a little effort, it can be shown that the characteristic polynomial of A is $(1-\lambda)(2+\lambda)^2$.)

The characteristic equation is

$$(1-\lambda)(2+\lambda)^2 = 0$$

A has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -2$.

Find eigenvectors:

For $\lambda_1 = 1$ $\quad A - 1I = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = x_3$
$x_2 = -x_3$
x_3 - free

Eigenvectors look like

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -2$

$$A - (-2)I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$
x_2 - free
$x_3 = 0$

Eigenvectors:

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

The algebraic multiplicity of $\lambda_2 = 2$ is two, but the geometric multiplicity is only one.

A is not diagonalizable.

Theorem (a second on diagonalizability)

Recall: (sec. 5.1) If λ_1 and λ_2 are distinct eigenvalues of a matrix, the corresponding eigenvectors are linearly independent.

Theorem: If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Note: This is a *sufficiency* condition, not a *necessity* condition. We've already seen a matrix with a repeated eigenvalue that was diagonalizable.

Theorem (a third on diagonalizability)

Theorem: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) The geometric multiplicity (dimension of the eigenspace) of λ_k is less than or equal to the algebraic multiplicity of λ_k.

(b) The matrix is diagonalizable if and only if the sum of the geometric multiplicities is n—i.e. the sum of dimensions of all eigenspaces is n so that there are n linearly independent eigenvectors.

(c) If A is diagonalizable, and $\mathcal{B}_k$ is a basis for the eigenspace for λ_k, then the collection (union) of bases $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

Remark: The union of the bases referred to in part (c) is called an **eigenvector basis** for $\mathbb{R}^n$. (Of course, one would need to reference the specific matrix.)

Example

Diagonalize the matrix if possible. $A = \begin{bmatrix} 5 & -6 \\ 4 & -5 \end{bmatrix}$.

① Find the eigenvalues.

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -6 \\ 4 & -5-\lambda \end{bmatrix}$$

$$= (5-\lambda)(-5-\lambda) + 24$$

$$= \lambda^2 - 25 + 24 \quad = \lambda^2 - 1$$

$$0 = \lambda^2 - 1 \implies \lambda_1 = 1 \quad \text{or} \quad \lambda_2 = -1$$

② Find eigenvectors

$\lambda_1 = 1$ $\quad A - 1I = \begin{bmatrix} 4 & -6 \\ 4 & -6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = \frac{3}{2}x_2$

x_2 - free

Eigenvectors $\quad \vec{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (eigenvector w/ $x_2 = 2$)

For $\lambda_2 = -1$

$A - (-1)I = \begin{bmatrix} 6 & -6 \\ 4 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = x_2$

x_2 - free

Eigenvectors $\quad \vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad$ set $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

③ Form matrix P

$$P = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

④ Find $D = P^{-1}AP$, $A = \begin{bmatrix} 5 & -6 \\ 4 & -5 \end{bmatrix}$ $\det(P) = 3 - 2 = 1$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 & -6 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Example Continued...

Find A^8 where $A = \begin{bmatrix} 5 & -6 \\ 4 & -5 \end{bmatrix}$.

* Recall: If A and B are similar matrices, then A^k and B^k are similar w/ the same similarity transformation matrix P.

We can use that

$$D^8 = P^{-1} A^8 P \implies A^8 = P D^8 P^{-1}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D^8 = \begin{bmatrix} 1^8 & 0 \\ 0 & (-1)^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^8 = P^{-1} D^8 P = P^{-1} I P = P^{-1} P$$

$$= I$$

Turns out

$A^{2k} = I$ for natural number k

$A^{2k+1} = A$ for " " " .

Section 6.1: Inner Product, Length, and Orthogonality

Recall: A vector $\mathbf{u}$ in $\mathbb{R}^n$ can be considered an $n \times 1$ matrix. It follows that $\mathbf{u}^T$ is a $1 \times n$ matrix.

$$\text{If}\quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \text{then} \quad \mathbf{u}^T = [u_1\ u_2\ \cdots\ u_n].$$

Definition of an Inner Product

Definition: For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ we define the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ (also called the **dot product**) by the **matrix product**

$$\underset{1\times n\ \ n\times 1}{\mathbf{u}^T\mathbf{v}} = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Remark: Note that this product produces a scalar. It is sometimes called a *scalar product*.

Theorem (Properties of the Inner Product)

We'll use the notations $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$ interchangeably.

Theorem: For $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and real scalar c

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(c) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$

(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$, with $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

The Norm

The property $\mathbf{u} \cdot \mathbf{u} \geq 0$ means that $\sqrt{\mathbf{u} \cdot \mathbf{u}}$ always exists as a real number.

Definition: The **norm** of the vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative number, denoted $\|\mathbf{v}\|$, defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

where $v_1, v_2, \ldots, v_n$ are the components of $\mathbf{v}$.

Remark: As a directed line segment, the norm is the same as the **length**.

Norm and Length

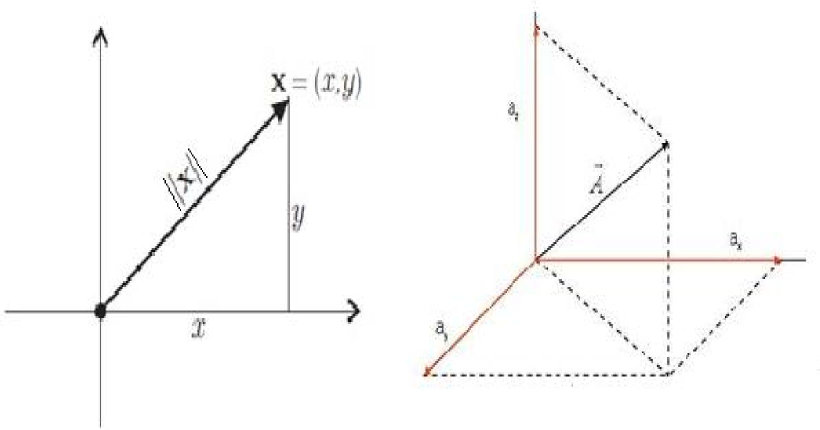

Figure: In $\mathbb{R}^2$ or $\mathbb{R}^3$, the norm corresponds to the classic geometric property of length.

Unit Vectors and Normalizing

Theorem: For vector $\mathbf{v}$ in $\mathbb{R}^n$ and scalar c

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Example: If $\mathbf{v}$ is a vector in $\mathbb{R}^4$ with norm $\|\mathbf{v}\| = 3$, then $-4\mathbf{v}$ is a vector in $\mathbb{R}^4$ with norm

$$\|-4\mathbf{v}\| = |-4|\,\|\mathbf{v}\| = 4 \cdot 3 = 12.$$

Unit Vectors and Normalizing

Definition: A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $\|\mathbf{u}\| = 1$ is called a **unit vector**.

Remark: Given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, we can obtain a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

This process, of dividing out the norm, is called **normalizing** the vector $\mathbf{v}$.

Example

Show that $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector.

Note $\dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\|\vec{v}\|}\vec{v}$ and $\|\vec{v}\| > 0$

$$\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \left| \frac{1}{\|\vec{v}\|} \right| \|\vec{v}\|$$

$$= \frac{1}{\|\vec{v}\|} \|\vec{v}\| = 1$$

Example

Find a unit vector in the direction of $\mathbf{v} = (1, 3, 2)$.

Let $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

$\|\vec{v}\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

$\vec{u} = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}$

Distance in $\mathbb{R}^n$

Definition: For vectors **u** and **v** in $\mathbb{R}^n$, the **distance between u and v** is denoted by

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}),$$

and is defined by

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Remark: This is the same as the traditional formula for distance used in $\mathbb{R}^2$ between points (x_0, y_0) and (x_1, y_1),

$$d = \sqrt{(y_1 - y_0)^2 + (x_1 - x_0)^2}.$$